Quantum Cosmological Gravitational Waves?

量子宇宙学引力波?

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Contents

目录

Abstract

摘要

General relativity and its cosmological solution predict the existence of tensor modes of perturbations evolving on top of our Friedman-Lemaître-Robertson-Walker expanding Universe. Being gauge invariant and not necessarily coupled to other quantum sources, they can be seen as representing pure gravity. Unambiguously showing they are indeed to be quantised would thus provide an unquestionable proof of the quantum nature of gravitation. This review will present a summary of the various theoretical issues that could lead to this conclusion.

广义相对论及其宇宙学解预言，在我们膨胀的弗里德曼-勒梅特-罗伯逊-沃尔克宇宙背景上演化的扰动存在张量模。张量模具有规范不变性，且不必与其他量子源耦合，可视为代表纯引力。明确证明张量模确实需要量子化，将为引力的量子本性提供毋庸置疑的证据。本综述将总结可推导出该结论的各类理论问题。

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Introduction

引言

Cosmology is a major player when it comes to quantum gravity effects. Indeed, on top of our Friedman-Lemaître-Robertson-Walker (FLRW) expanding Universe, one expects various modes of perturbations to be present, whose classical occurrence is believed to result from initial quantum vacuum fluctuations. In the usual linear formalism [1, 2], using the FLRW underlying symmetry group (isotropy and homogeneity), they can be categorised into three components, namely, scalars, vectors and tensors. At this order, upon which we focus attention below, these components decouple. In a different situation with a background endowed with other symmetries, perturbations can still be expanded in the relevant representations of the associated group; they also naturally decouple at linear order (see, e.g., Ref. [3] for Bianchi I).

在量子引力效应研究领域，宇宙学是一门核心学科。在弗里德曼-勒梅特-罗伯逊-沃尔克 (FLRW) 膨胀宇宙中，预计存在多种扰动模式，一般认为这些扰动的经典起源是初始量子真空涨落。在常规线性形式体系 [1, 2] 中，借助 FLRW 背景的对称群 (各向同性与均匀性)，这些扰动可以分为三类: 标量扰动、矢量扰动和张量扰动。本文我们聚焦的线性阶层面上，这三类分量是解耦的。对于拥有其他对称性背景的不同情形，扰动仍可以按照对应对称群的相关表示展开，并且在线性阶自然解耦 (例如参见比安基 I 型宇宙的相关研究 [3])。

Scalar modes, detected long ago in the cosmic microwave background, initiating large-scale structure formation, are distributed in a way that is compatible with quantum vacuum fluctuations in the very early times, often during a phase of inflation. This can be seen as requiring quantisation of gravity, and although many authors consider it does, others argue that gauge issues and coupling with matter render the conclusion not as clear as one would wish.

标量模式早已在宇宙微波背景中被探测到，它引发了大尺度结构的形成，其分布规律与极早期宇宙 (通常处于暴胀阶段) 的量子真空涨落相容。这一观测结果通常被认为要求引力量子化，尽管许多学者认同这一观点，但也有观点认为规范问题以及引力与物质的耦合让这一结论并不如预期那般清晰。

In an ever-expanding FLRW universe with dynamics driven by GR or any local theory of gravity, with no specific source in the matter fields to induce them, vector perturbations are expected to have decayed long ago so as to be mostly undetectable now [4]. One of the above hypothesis needs to be invalidated to potentially render them cosmologically relevant. Non local theories are expected to yield conclusions similar to local ones [5]. A contracting phase in the universe as implemented in bouncing models [6, 7] can lead to some increase of vector modes [8] which are however limited if produced by means of some coupling with scalar modes initially set to quantum vacuum fluctuations [9], leading to the conclusion that bouncing models are generally stable under vector perturbations. For fully quantum cosmological models however, the situation may not be as clear [10]. In any case, the question of their quantum origin would lead to similar doubts regarding the quantumness of gravity itself; they are conveniently ignored in most studies and likewise in the present review.

在由广义相对论或任意局域引力理论驱动动力学的持续膨胀 FLRW 宇宙中，如果物质场没有特殊源激发矢量扰动，那么矢量扰动早已发生衰减，如今几乎无法被探测到 [4]。只有打破上述某一个假设，矢量扰动才可能具有宇宙学相关性。一般认为非局域理论也会得到和局域理论类似的结论 [5]。bounce 宇宙模型 [6, 7] 中的收缩阶段会让矢量扰动幅度有所增加 [8]，但如果矢量扰动是由初始处于量子真空涨落的标量模式通过某种耦合产生的，其增幅会受到限制，因此得出结论:bounce 模型在矢量扰动下通常是稳定的。不过，在全量子宇宙学模型中，情况可能并非如此清晰 [10]。无论如何，关于矢量扰动量子起源的问题，也会给引力本身的量子属性带来类似的疑问，因此大多数研究都将其忽略，本文综述同样如此处理。

Finally, one is left with the tensor modes, which are gauge invariant and with no obvious coupling to other quantum sources. General relativity (GR) applied to primordial cosmology shows their dynamics to be that of two time-dependent massive scalar fields; most models then demand they should be quantised and set in a vacuum state. The observation of their resulting properties in the absence of quantum anisotropic pressure, jointly with those of the scalar modes, could provide an unambiguous and thus indisputable hint that gravitation itself should acquire the status of a quantum theory.

最后剩下的是张量模式，它是规范不变的，也不存在与其他量子源的明显耦合。原初宇宙学中的广义相对论 (GR) 表明，张量模式的动力学等价于两个含时有质量标量场的动力学；大多数模型都要求对张量模式量子化，并将其初始状态设为真空态。在不存在量子各向异性压强的情况下观测张量模式的导出性质，结合标量模式的观测结果，能够给出明确且无可争议的线索，证明引力本身应当是一个量子理论。

Tensor Modes in General Relativistic Cosmology

广义相对论宇宙学中的张量模式

Before focusing on the quantum features expected from gravitational waves, let us briefly recap the underlying classical theory. The starting point of our discussion is the FLRW background universe, defined by its scale factor function $a(\eta)$ depending on the (conformal) time η and spatial 3D metric γ_{ij} , with tensorial perturbations h_{ij} . In that case, ignoring both scalar and vector modes which are not the subject of this analysis, one sets the metric as

在讨论引力波的量子特性之前，我们先简要回顾其基础经典理论。我们的讨论起点是 FLRW 背景宇宙，由依赖共形时间 η 的标度因子函数 $a(\eta)$ 、三维空间度规 γ_{ij} 以及张量微扰 h_{ij} 定义。这种情况下，我们忽略不属于本文分析对象的标量模式和矢量模式，将度规写为

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\eta)\left[-\mathrm{d}\eta^2 + \left(\gamma_{ij} + h_{ij}\right)\mathrm{d}x^i\,\mathrm{d}x^j\right], \tag{1}$$

(we use units such that the velocity of light is $c = 1$) and with h_{ij} assumed transverse and traceless, i.e.,

(我们采用的单位制中光速为 $c = 1$)，且假设 h_{ij} 是横波无迹的，即

$$D^i h_{ij} = 0 \quad \text{and} \quad h^i{}_i = \gamma^{ij} h_{ij} = 0,$$

the 3D covariant derivative D^i being derived from the corresponding metric γ_{ij} . We note $\mathcal{H} = a'/a$ the conformal Hubble function, a prime standing for differentiation with respect to conformal time $(' := d/d\eta)$, and $\mathcal{K}$ the spatial curvature associated with the background metric $\gamma_i j$. In appropriate units for the comoving coordinates $x^i, \mathcal{K}$ can be scaled to $\mathcal{K} = 0, \pm 1$. For most of the practical applications we shall deal with in this review, we shall consider the simplest, and inflation-motivated, flat case with $\mathcal{K} = 0$. The equation of motion for h_{ij} is found to be

三维协变导数 D^i 由对应度规 γ_{ij} 导出。我们记 $\mathcal{H} = a'/a$ 为共形哈勃函数，撇号表示对共形时间 $(' := d/d\eta)$ 求导，$\mathcal{K}$ 为背景度规 $\gamma_i j$ 对应的空间曲率。在合适的单位下，共动坐标 $x^i, \mathcal{K}$ 可以标度为 $\mathcal{K} = 0, \pm 1$ 。在本综述将要讨论的绝大多数实际应用中，我们考虑受暴胀理论启发的最简单平坦情况，即满足 $\mathcal{K} = 0$ 。可得 h_{ij} 的运动方程为

$$h''_{ij} + 2\mathcal{H}h'_{ij} + (2\mathcal{K} - \Delta)\, h_{ij} = 8\pi G_{\mathrm{N}} a^2 p \pi_{ij}, \tag{2}$$

where $\Delta = \gamma^{ij}\partial_i\partial_j, p$ is the background pressure and π_{ij} the anisotropic stress and G_N the Newton gravitational constant. For many of the known components of matter, it is vanishing (see, however, e.g., Ref. [11,12] and references therein), and we shall make the assumption that $\pi_{ij} = 0$ from now on.

其中 $\Delta = \gamma^{ij}\partial_i\partial_j, p$ 是背景压强，π_{ij} 是各向异性应力，G_N 是牛顿引力常数。对于大多数已知物质组分，各向异性应力为零 (不过参见例如文献 [11,12] 及其中引用)，因此从现在开始我们假设 $\pi_{ij} = 0$ 。

In what follows, we set $\mathcal{K} \to 0$ and thus identify the background spatial metric $\gamma^{ij} \to \delta^{ij}$ as the 3D curvature has been measured to be vanishingly small. Technically, considering a non-vanishing curvature merely amounts to changing the spectrum (and eigenfunctions) of the Laplace-Beltrami operator used below for the mode decomposition [13], so that the calculations and discussions presented below can be generalised in a straightforward way if applied to epochs in which the assumption $\mathcal{K} = 0$ may not be valid.

在下文中，我们令 $\mathcal{K} \to 0$ ，并由此确定背景空间度规 $\gamma^{ij} \to \delta^{ij}$ ，因为观测已经测得空间曲率几乎为零。从技术上讲，考虑非零曲率仅仅相当于改变下文模式分解所用拉普拉斯-贝尔特拉米算子的谱 (和本征函数)[13]，因此如果将下文的计算和讨论应用到 $\mathcal{K} = 0$ 假设不成立的宇宙学时期，也可以很方便地推广。

Let us thus first decompose the tensor perturbations in Fourier modes through

那么我们首先将张量微扰按傅里叶模式分解如下

$$h_{ij}(\mathbf{x}, \eta) = \sqrt{32\pi G_{\mathrm{N}}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2} a(\eta)} w_{ij}(\mathbf{k}, \eta)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3}$$

the numerical factor $\sqrt{32\pi G_{\mathrm{N}}}$ is included here for later convenience. To ensure $w^\star_{ij}(-\mathbf{k}, \eta) = w_{ij}(\mathbf{k}, \eta)$ to ensure $h_{ij} \in \mathbb{R}$, so that, from Eq. (2), a given mode satisfies

此处加入数值因子 $\sqrt{32\pi G_{\mathrm{N}}}$ 是为了后续计算方便。为满足 $w^\star_{ij}(-\mathbf{k}, \eta) = w_{ij}(\mathbf{k}, \eta)$ 以保证 $h_{ij} \in \mathbb{R}$ ，因此由式 (2) 可得任意模式满足

$$w''_{ij} + \omega_k^2 w_{ij} = 0 \tag{4}$$

where we defined the module $k := |\mathbf{k}| \geq 0$ and the time-varying frequency

其中我们定义了波数 $k := |\mathbf{k}| \geq 0$ 以及随时间变化的频率

$$\omega_k^2 = k^2 - \frac{a''}{a}. \tag{5}$$

At this point, one notes that whenever the scale factor behaves as a power law of the conformal time $a(\eta) \propto |\eta|^\alpha$, then $a''/a = \alpha(\alpha-1)/\eta^{-2} = (\alpha-1)\mathcal{H}^2/\alpha$. Here and in what follows, we write the absolute value of the conformal time, as it is negative in many situations, in particular during inflation. This in particular encompasses the cases of cosmological interest where a single fluid dominates the Friedmann dynamics, as well as the de Sitter inflationary expansion. The condition $k^2 \gg |a''/a|$ then becomes $k\mathcal{H}^{-1} \propto k|\eta| \ll 1$, so that, in terms of the physical wavelength $\lambda \propto a/k$, one has $\lambda \ll H^{-1}$: such a mode, much smaller than the Hubble scale H^{-1} , is said to be sub-Hubble. Conversely, modes with $k^2 \ll |a''/a|$ are called super-Hubble.

此时可以发现，只要标度因子是共形时间 $a(\eta) \propto |\eta|^\alpha$ 的幂律形式，就有 $a''/a = \alpha(\alpha-1)/\eta^{-2} = (\alpha-1)\mathcal{H}^2/\alpha$ 。本文及后续推导中，我们对共形时间取绝对值，因为共形时间在很多场景下为负，暴胀时期尤其如此。这一形式尤其涵盖了单一流体主导弗里德曼动力学，以及德西特暴胀膨胀这些宇宙学感兴趣的情形。此时条件 $k^2 \gg |a''/a|$ 变为 $k\mathcal{H}^{-1} \propto k|\eta| \ll 1$ ，因此对物理波长 $\lambda \propto a/k$ 可得 $\lambda \ll H^{-1}$: 这种远小于哈勃尺度 H^{-1} 的模式称为亚哈勃模式。反之，满足 $k^2 \ll |a''/a|$ 的模式称为超哈勃模式。

Let us temporarily restrict attention to a sub-Hubble mode $k^2 \gg |a''/a|$. Equation (4) then simplifies to $w''_{ij} + k^2 w_{ij} = 0$, whose solution reads $w_{ij} = \alpha_{ij}\exp(\pm ik\eta)$. For a mode propagating in the $+x^3$ -direction, this yields $h_{ij} = \alpha_{ij}\exp\left[\pm ik\left(x^3-\eta\right)\right]/a(\eta)$. The first constraint, namely, $\partial^i h_{ij} = 0$, implies $k^i\alpha_{ij} = k\alpha_{zj} = 0$, so that for $k \neq 0$, one is left with α_{11}, α_{12} and α_{22} as the only non-vanishing components, the symmetries of w_{ij} being identical to those of h_{ij} . The second constraint, $h^i{}_i = 0$, translates into $\alpha_{22} = -\alpha_{11}$, so the mode has only two independent degrees of freedom. The matrix α_{ij} can be rewritten explicitly as

我们暂时将注意力限制在亚哈勃模式 $k^2 \gg |a''/a|$ 上。此时方程 (4) 简化为 $w''_{ij} + k^2 w_{ij} = 0$ ，其解为 $w_{ij} = \alpha_{ij}\exp(\pm ik\eta)$ 。对沿 $+x^3$ 方向传播的模式，可得 $h_{ij} = \alpha_{ij}\exp\left[\pm ik\left(x^3-\eta\right)\right]/a(\eta)$ 。第一个约束即 $\partial^i h_{ij} = 0$ ，推得 $k^i\alpha_{ij} = k\alpha_{zj} = 0$ ，因此当 $k \neq 0$ 时，仅剩下 α_{11}, α_{12} 和 α_{22} 为非零分量，且 w_{ij} 的对称性与 h_{ij} 完全一致。第二个约束 $h^i{}_i = 0$ 转化为 $\alpha_{22} = -\alpha_{11}$ ，因此该模式仅存在两个独立自由度。矩阵 α_{ij} 可以显式改写为

$$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{12} & -\alpha_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=\sqrt{2}P^+_{ij}}\alpha_{11} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=\sqrt{2}P^\times_{ij}}\alpha_{12}. \tag{6}$$

The matrices P^+_{ij} and $P^\times_{ij}$ represent the two polarisations of the gravitational wave, whose associated tensor perturbations read

矩阵 P_{ij}^{+} 和 $P_{ij}^{\times}$ 代表引力波的两种偏振，对应的张量扰动为

$$h_{ij}(\mathbf{x},\eta) = h_{\times}(t-z)\,P_{ij}^{\times} + h_{+}(t-z)\,P_{ij}^{+}, \tag{7}$$

with $\{t,x,y,z\} = \left\{an, ax^1, ax^2, ax^3\right\}$ as the physical coordinates.

其中 $\{t,x,y,z\} = \left\{an, ax^1, ax^2, ax^3\right\}$ 为物理坐标。

Consider a test particle following the trajectory of affine parameter λ , i.e., $x^{\mu}(\lambda)$, and initially at rest in the TT-frame where the metric has the form (1) with h_{ij} given by Eq. (7), namely, assuming the scale factor a to be constant during the passing of the wave,

考虑一个遵循仿射参数 λ 轨迹的试探粒子，即满足 $x^{\mu}(\lambda)$ ，且初始时在 TT 坐标系中静止，该坐标系下度规具有形式 (1)，其中 h_{ij} 由式 (7) 给出，也就是假设引力波经过过程中标度因子 a 保持恒定，

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + [1+h_{+}(z-t)]\,\mathrm{d}x^2 + [1-h_{+}(z-t)]\,\mathrm{d}y^2 + 2h_{\times}(z-t)\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}z^2.$$

(8)

From Eq. (8), one can evaluate the connections while the wave passes, and it turns out that $\Gamma^{i}_{\eta\eta} = 0$, so that the motion of a particle following a geodesic is unaltered as it moves with the reference frame: it appears at rest at all times. It is therefore not possible to detect a gravitational wave using a single particle.

由式 (8)，我们可以计算引力波经过时的联络，结果得到 $\Gamma^{i}_{\eta\eta} = 0$ ，因此沿测地线运动的粒子随参考系运动时运动状态不会改变: 它始终看起来是静止的。因此仅用单个粒子无法探测引力波。

Writing the line element as $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\ell^2$, we consider two particles located on the TT- x axis (i.e., $y = z = 0$) with coordinates x and $x + \Delta x$. Their proper distance is obtained from Eq. (8): the relation $\mathrm{d}\ell_x = \sqrt{1+\alpha_{+}(t)}\mathrm{d}x \simeq \left[1+\frac{1}{2}h_{+}(t)\right]\mathrm{d}x$ can be integrated to yield $\Delta\ell_x = \left[1+\frac{1}{2}h_{+}(t)\right]\Delta x$. Similarly, considering two particles lying along the y axis, one obtains $\Delta\ell_y = \left[1-\frac{1}{2}h_{+}(t)\right]\Delta y$, so that as the separation along one direction is elongated, the other is compressed, and vice versa. A similar calculation on particles set on the $y = \pm x$ lines permits to visualise the effect of the $\alpha_{\times}$ polarisation. Setting our test particles along a ring in the (x,y) plane, such as shown in Fig. 1, one gets the $+$ and $\times$ shapes as the wave propagates in the z -direction, hence the names of the polarisation modes.

将线元写为 $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\ell^2$ ，我们考虑位于 TT- x 轴上的两个粒子 (即 $y = z = 0$)，其坐标分别为 x 和 $x + \Delta x$ 。可由式 (8) 得到它们的固有距离: 对关系式 $\mathrm{d}\ell_x = \sqrt{1+\alpha_{+}(t)}\mathrm{d}x \simeq \left[1+\frac{1}{2}h_{+}(t)\right]\mathrm{d}x$ 积分可得 $\Delta\ell_x = \left[1+\frac{1}{2}h_{+}(t)\right]\Delta x$ 。同理，考虑沿 y 轴分布的两个粒子，可得 $\Delta\ell_y = \left[1-\frac{1}{2}h_{+}(t)\right]\Delta y$ ，因此当一个方向的间距被拉伸时，另一个方向会被压缩，反之亦然。对位于 $y = \pm x$ 线上的粒子做类似计算，可以直观得到 $\alpha_{\times}$ 偏振的效应。将我们的测试粒子布置在 (x,y) 平面的一个环上，如图 1 所示，当波沿 z 方向传播时，会得到 $+$ 和 $\times$ 形变，这也是偏振模式名称的来源。

For a general wave vector $\mathbf{k} = k\mathbf{n}$ in the arbitrary direction parametrised by the angles φ and θ (see Fig. 2), namely, $\mathbf{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$, one sets

对于由角度 φ 和 θ 参数化的任意方向上的一般波矢 $\mathbf{k} = k\mathbf{n}$ (参见图 2)，即 $\mathbf{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$，我们设定

$$w_{ij}(\mathbf{k},\eta) = \sum_{\lambda=+,\times} P_{ij}^{(\lambda)}(\mathbf{n})\, f_\lambda(\mathbf{k},\eta), \tag{9}$$

with $P_{ij}^{(\lambda)}(\mathbf{n})$ the polarisation tensors and f_λ the associated functions solutions of the mode equation (4). Figure 2 shows the vectors $\mathbf{e}_a$ $(a = 1, 2)$ generating the plane orthogonal to the direction of propagation. Defined through

其中 $P_{ij}^{(\lambda)}(\mathbf{n})$ 为偏振张量，f_λ 为模式方程 (4) 的对应函数解。图 2 给出了生成垂直于传播方向平面的矢量 $\mathbf{e}_a$ $(a = 1, 2)$，其定义为

$$\mathbf{e}_1 = -\frac{1}{\sin\theta}\frac{\partial\mathbf{n}}{\partial\varphi} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_2 = \frac{\partial\mathbf{n}}{\partial\theta} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix},$$

so that $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$, they satisfy $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$ and $\mathbf{n} \cdot \mathbf{e}_a = 0$. Demanding h_{ij} to be transverse and traceless translates into

因此有 $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$，它们满足 $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$ 和 $\mathbf{n} \cdot \mathbf{e}_a = 0$。要求 h_{ij} 为横波且无迹等价于

$$k^i P_{ij}^{(\lambda)} = 0, \text{ and } P_{ij}^{(\lambda)}\delta^{ij} = 0, \tag{10}$$

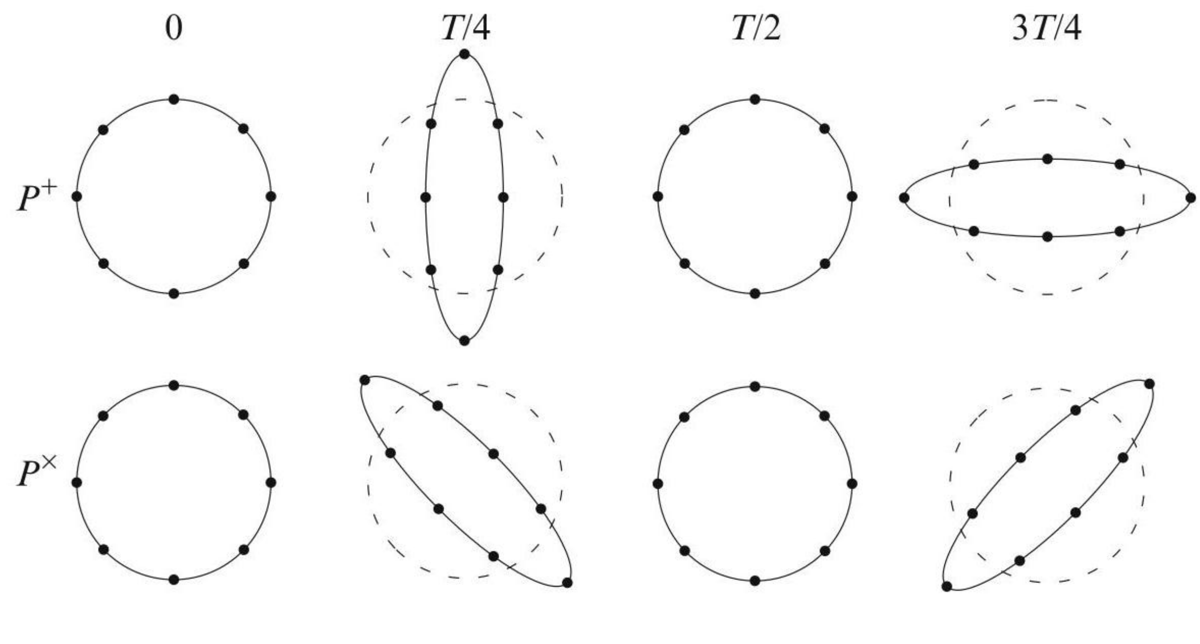


Fig. 1 Effect of a gravitational wave mode P^+ or $P^\times$ as it passes through a ring of test particles, producing "+" or " $\times$ " shapes as time goes through a full period of the wave: starting with an initially circular ring at

$t = 0$, its shape is modified and shown here for different values of time, namely, $T/4, T/2$ and $3T/4$ for a period $T = 2\pi/k$

图 1 引力波模式 P^+ 或 $P^\times$ 穿过测试粒子环的效应: 在波的一个完整周期内，会产生"+" 或" $\times$ " 形变: 初始为圆形环，对应 $t = 0$ 时刻，图中展示了一个周期 $T = 2\pi/k$ 内不同时刻 (即 $T/4, T/2$ 和 $3T/4$) 的形状变化

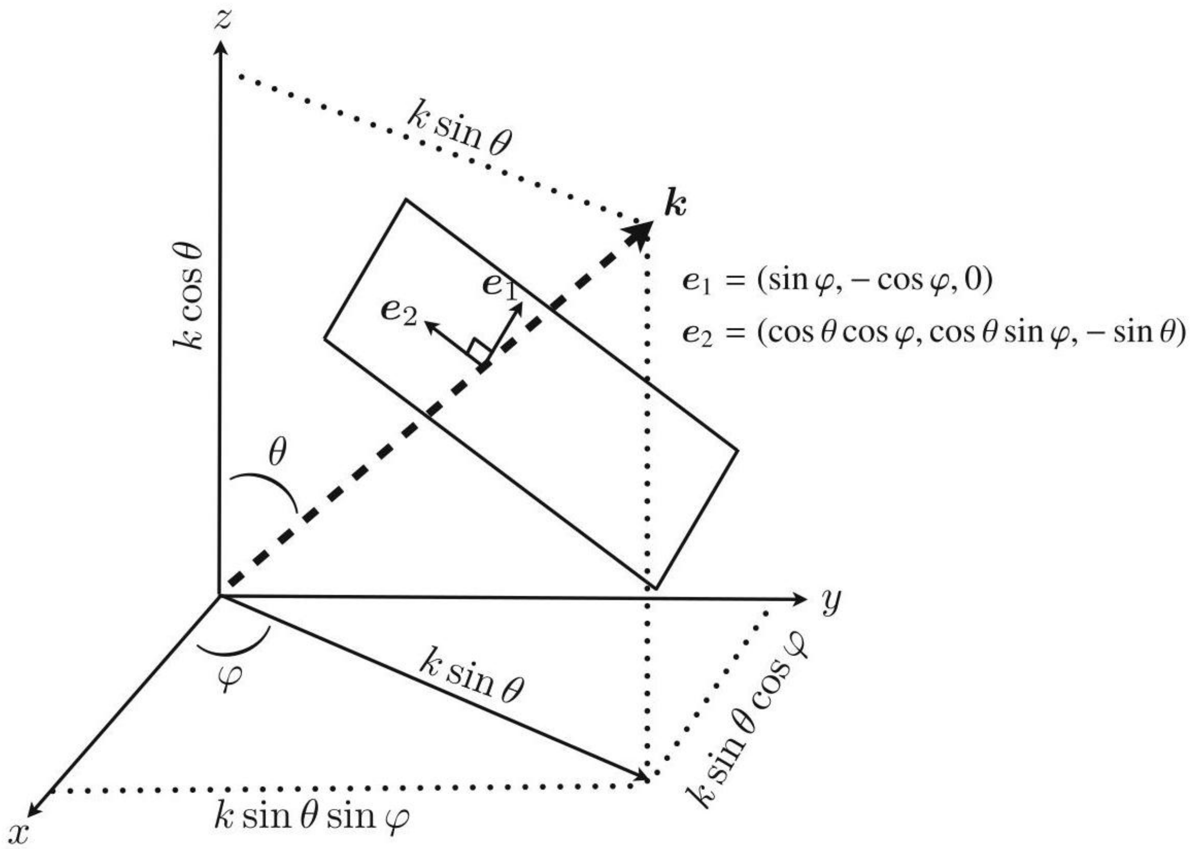


Fig. 2 Definition of the dyad $\mathbf{e}_a$ $(a = 1, 2)$ in the plane orthogonal to the arbitrary direction $\mathbf{k}$

图 2 垂直于任意方向 $\mathbf{k}$ 平面内双矢量基 $\mathbf{e}_a$ $(a = 1, 2)$ 的定义

and one can check that the choice

并且可以验证该选择

$$P^+_{ij} = \frac{1}{\sqrt{2}}\left[(\mathbf{e}_2)_i(\mathbf{e}_2)_j - (\mathbf{e}_1)_i(\mathbf{e}_1)_j\right] \text{ and } P^\times_{ij} = -\frac{1}{\sqrt{2}}\left[(\mathbf{e}_1)_i(\mathbf{e}_2)_j + (\mathbf{e}_2)_i(\mathbf{e}_1)_j\right]$$

(11)

satisfies all the constraints (10); they reduce to those appearing in Eq. (6) for $\mathbf{k} = kz$ (choosing $\varphi \to 0$ or $\varphi \to \pi$ as it is then undetermined). One can check straightforwardly that the relations

满足所有约束条件 (10)；当 $\mathbf{k} = kz$ 时，这些约束退化为式 (6) 中的形式 (此时选择 $\varphi \to 0$ 或 $\varphi \to \pi$ 都不影响，因为该量未被确定)。可以直接验证关系

$$P_j^{i+}(\mathbf{n})\,P_i^{j+}(\mathbf{n}) = P_j^{i\times}(\mathbf{n})\,P_i^{j\times}(\mathbf{n}) = 1 \quad \text{and} \quad P_j^{i+}(\mathbf{n})\,P_i^{j\times}(\mathbf{n}) = 0 \tag{12}$$

hold.

成立。

Let us note at this point that the transformation $\mathbf{n} \to -\mathbf{n}$, which amounts to $(\theta, \varphi) \to (\pi - \theta, \varphi + \pi)$, implies $\mathbf{e}_1 \to -\mathbf{e_1}$ and $\mathbf{e}_2 \to \mathbf{e_2}$, so that

在此我们需要指出，变换 $\mathbf{n} \to -\mathbf{n}$ 等价于 $(\theta, \varphi) \to (\pi - \theta, \varphi + \pi)$ ，可推导出 $\mathbf{e}_1 \to -\mathbf{e_1}$ 和 $\mathbf{e}_2 \to \mathbf{e_2}$ ，因此

$$P_{ij}^{+}(-\mathbf{n}) = P_{ij}^{+}(\mathbf{n}) \quad \text{and} \quad P_{ij}^{\times}(-\mathbf{n}) = -P_{ij}^{\times}(\mathbf{n})\,. \tag{13}$$

From Eq. (9) and the reality condition below Eq. (3), one then finds that

根据式 (9) 和式 (3) 下方的实条件，我们可以得到

$$f_{+}^{\star}(-\mathbf{k}, \eta) = f_{+}(\mathbf{k}, \eta) \quad \text{and} \quad f_{\times}^{\star}(-\mathbf{k}, \eta) = -f_{\times}(\mathbf{k}, \eta)\,,$$

the extra minus sign in the cross-polarisation reflecting the fact that the gravitational wave transforms according to a spin-2 representation and not as a scalar. This sign is however inconvenient as it requires the functions f_λ to explicitly depend on the direction of propagation of the gravitational wave.

交叉极化中的额外负号反映了引力波按自旋 2 表示而非标量变换这一事实。但这个负号会带来不便，因为它要求函数 f_λ 显式依赖于引力波的传播方向。

This issue is solved by considering another basis instead of the + and × polarisations:

这个问题可以通过引入另一个基来解决，替换原有的 + 和 × 极化:

$$\varepsilon_{ij}^{\pm} \;:=\; \frac{1}{\sqrt{2}}\left(P_{ij}^{+} \pm i P_{ij}^{\times}\right), \tag{14}$$

resulting in the new expansion

从而得到新的展开式

$$w_{ij}(\mathbf{k}, \eta) = \sum_{\lambda=\pm} \varepsilon_{ij}^{(\lambda)}(\mathbf{n})\,\mu_\lambda(\mathbf{k}, \eta)\,, \tag{15}$$

where now one recovers the usual reality conditions in the form

此时我们可以得到常见形式的实条件

$$\mu_{\pm}^{\star}(-\mathbf{k},\eta)=\mu_{\pm}(\mathbf{k},\eta)\,, \tag{16}$$

because $\left[\varepsilon_{ij}^{\pm}(-\mathbf{n})\right]^{\star}=\varepsilon_{ij}^{\pm}(\mathbf{n})$. Note also that the orthogonality relations become $\varepsilon^{i\pm}(\mathbf{n})\,\varepsilon_{i}^{j\mp}(\mathbf{n})=1$ and $\varepsilon^{i\pm}(\mathbf{n})\,\varepsilon^{j\pm}(\mathbf{n})=0$ and that the coefficients of the expansion are related via

原因是 $\left[\varepsilon_{ij}^{\pm}(-\mathbf{n})\right]^{\star}=\varepsilon_{ij}^{\pm}(\mathbf{n})$。另外还需注意，正交关系变为 $\varepsilon^{i\pm}(\mathbf{n})\,\varepsilon_{i}^{j\mp}(\mathbf{n})=1$ 和 $\varepsilon^{i\pm}(\mathbf{n})\,\varepsilon^{j\pm}(\mathbf{n})=0$ ，且展开系数满足关系

$$\mu_{\pm}(\mathbf{k},\eta)=\frac{1}{\sqrt{2}}\left[f_{+}(\mathbf{k},\eta)\mp i f_{-}(\mathbf{k},\eta)\right]. \tag{17}$$

Performing a rotation in the plane orthogonal to $\mathbf{n}$ by an angle α amounts to rotating $\mathbf{e}_a$ through

在垂直于 $\mathbf{n}$ 的平面内旋转角度 α 等价于对 $\mathbf{e}_a$ 做如下旋转

$$\begin{cases}\mathbf{e}_1\rightarrow\mathbf{e}_1\cos\alpha-\mathbf{e}_2\sin\alpha\\ \mathbf{e}_2\rightarrow\mathbf{e}_1\sin\alpha+\mathbf{e}_2\cos\alpha\end{cases}$$

and one can check explicitly that the new polarisations transform according to

我们可以直接验证，新的极化满足变换规则

$$\varepsilon_{ij}^{\pm}\rightarrow e^{\pm 2i\alpha}\varepsilon_{ij}^{\pm} \tag{18}$$

i.e., they transform as tensors with helicity ± 2 and are therefore referred to as the helicity basis. Gathering all the above, one finds that Eq. (15) permits to show that, in general, both the modes μ_+ and μ_- satisfy the same equation of motion, which is nothing but Eq. (4) with the replacement $w_{ij}\rightarrow\mu_{\pm}$.

即它们按螺旋度为 ± 2 的张量变换，因此被称为螺旋度基。综上，式 (15) 表明一般情况下，模式 μ_+ 和 μ_- 都满足相同的运动方程，该方程就是将式 (4) 替换为 $w_{ij}\rightarrow\mu_{\pm}$ 后得到的结果。

This can be derived directly from Eq. (4) using the expansion on the helicity basis or going back to the Einstein-Hilbert action and performing an expansion in powers of h_{ij}

这一结果可以直接从式 (4) 出发，利用螺旋度基展开，或回到爱因斯坦-希尔伯特作用量按 h_{ij} 的幂次展开得到

$$S_{\mathrm{EH}}=\frac{1}{16\pi G_{\mathrm{N}}}\int \mathrm{d}^4x\sqrt{-g}R=\frac{1}{16\pi G_{\mathrm{N}}}\int \mathrm{d}^4x\sqrt{-\left[g^{(0)}+g^{(2)}\right]}\left[R^{(0)}+R^{(2)}\right]+\cdots,$$

where the dots represent higher-order terms and the determinant is expanded as the exponent of the trace of a logarithm $g=\det\left(g_{\mu\nu}\right)=\det\left(g_{\mu\nu}^{(0)}\right)\det\left(\delta_{\nu}^{\mu}+h_{\nu}^{\mu}\right)=a^4\left[1-\frac{1}{2}h_j^i h_i^j+\mathcal{O}\left(h^3\right)\right]$ and the first term vanishes due to the traceless condition, while the contribution from $R^{(1)}$ vanishes identically if we assume the background to satisfy the equation of motion. The resulting action at second-order reads

其中省略号代表高阶项，行列式按对数迹的指数展开 $g = \det\left(g_{\mu\nu}\right) = \det\left(g^{(0)}_{\mu\nu}\right)\det\left(\delta^{\mu}_{v} + h^{\mu}_{v}\right) = a^4\left[1 - \frac{1}{2}h^i_j h^j_i + \mathcal{O}\left(h^3\right)\right]$，一阶项因无迹条件消失，而若假设背景满足运动方程，$R^{(1)}$ 的贡献恒为零。二阶作用量可写为

$$\delta^{(2)}S_{\mathrm{T}} = \frac{1}{64\pi G_{\mathrm{N}}}\int a^2(\eta)\left[\frac{\partial h^i}{\partial\eta}\frac{\partial h^j_i}{\partial\eta} - \left(\partial_k h^i{}_j\right)\partial^k h^j\right]\mathrm{d}^4x. \tag{19}$$

Plugging the expansion (3) and the definition (15) into the action (19) leads to

将展开式 (3) 和定义式 (15) 代入作用量 (19)，可得

$$\delta^{(2)}S_{\mathrm{T}} = \int \mathrm{d}\eta \sum_{\lambda=\pm}\frac{1}{2}\int \mathrm{d}^3\mathbf{k}\left[\left(\mu'^{\star}_{\lambda} - \mathcal{H}\mu^{\star}_{\lambda}\right)\left(\mu'_{\lambda} - \mathcal{H}\mu_{\lambda}\right) - \mathbf{k}^2\mu^{\star}_{\lambda}\mu_{\lambda}\right]. \tag{20}$$

Upon integrating the $\mathcal{H}\left(\mu^{\star}_{\lambda}\mu_{\lambda}\right)'$ by parts, and using Parseval theorem to revert to real space, we get

对 $\mathcal{H}\left(\mu^{\star}_{\lambda}\mu_{\lambda}\right)'$ 做分部积分，再利用帕塞瓦尔定理回到实空间，我们得到

$$\delta^{(2)}S_{\mathrm{T}} = \int \mathrm{d}\eta \sum_{\lambda=\pm}\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\sqrt{\gamma}\left[\left(\mu'_{\lambda}\right)^2 - \gamma^{ij}\partial_i\mu_{\lambda}\partial_j\mu_{\lambda} + \frac{a''}{a}\mu^2_{\lambda}\right], \tag{21}$$

where we wrote $\mu_{\lambda} = \mu_{\lambda}(\mathbf{x},\eta)$ the inverse Fourier transform of $\mu_{\lambda}(\mathbf{k},\eta)$. This is the action for two independent scalar fields μ_+ and μ_- , with identical time-varying masses. One can check that the Euler-Lagrange equation for the action (21) gives back Eq. (4) for both polarisations. The form (20) allows for straightforward quantisation of the gravitational field as a collection of parametric oscillators, which is the subject of the following section.

其中我们将 $\mu_{\lambda} = \mu_{\lambda}(\mathbf{x},\eta)$ 写为 $\mu_{\lambda}(\mathbf{k},\eta)$ 的傅里叶逆变换。这是两个独立标量场 μ_+ 和 μ_- 的作用量，二者具有相同的时变质量。可以验证，作用量 (21) 的欧拉-拉格朗日方程对两种偏振都还原出式 (4)。形式 (20) 可直接将引力场量子化为一组参量振荡器，这是下一节的主题。

Quantisation and Time Development

量子化与时间演化

Historical Perspective

历史视角

Parker, in Ref. [14], was the first to use the above separation of the gravitational wave field into two minimally coupled scalar fields as a simpler route to quan-tisation, although previous works on (quantum) fields in curved spacetime had already identified the crucial prediction of (vacuum) amplification powered by the expansion of the Universe, including for gravitational waves. Particle creation following a change in boundary conditions of a system was shown in Ref. [15], but creation powered by an expanding Universe was first demonstrated by Parker in his seminal articles [16-18]. However, it was argued that massless non-zero spin fields, including gravitational waves, had to be conformally coupled to gravity so that no particle creation could occur. The production of gravitons, particles associated with gravitational waves, was studied in anisotropic universes in Ref. [19] and hinted at in Ref. [20], but Grishchuk [21] was the first to lift the misunderstanding and to compute the ensuing gravitational wave amplification in an isotropic expanding universe. Despite the use of a classical treatment, the corresponding quantum particle pair creation was noted, and the existence of a primordial gravitational wave background put forward. Several authors then attempted to compute the spectrum of this background based on spontaneous pair creation from the vacuum still using a classical treatment and different initial conditions and renormalisation procedures, e.g., in Refs. [22-24]. Finally, in Ref. [24], graviton production due to an early de Sitter phase of expansion, not yet called inflation, was considered with the Bunch-Davies vacuum [25] providing the relevant initial conditions.

尽管此前关于弯曲时空下 (量子) 场的研究已经得出了关键结论: 宇宙膨胀会驱动 (真空) 放大，这一效应同样适用于引力波，但帕克在文献 [14] 中首次将引力波场拆分为两个最小耦合标量场，以此作为更简便的量子化途径。文献 [15] 证明了系统边界条件变化会引发粒子产生，而宇宙膨胀驱动的粒子产生最早由帕克在其开创性文献 [16-18] 中证明。不过曾有观点认为，包括引力波在内的无质量非零自旋场必须与引力共形耦合，因此不会产生粒子。文献 [19] 在各向异性宇宙中研究了引力子 (即与引力波对应的粒子) 的产生，文献 [20] 也给出了相关提示，但格里什丘克 [21] 第一个纠正了这一误解，并计算了各向同性膨胀宇宙中引力波的放大效应。尽管格里什丘克采用经典处理，但他仍指出了对应的量子粒子对产生，并提出原初引力波背景是存在的。随后多位学者仍采用经典处理，结合不同初始条件和重整化方案，尝试基于真空自发对产生计算该背景的谱，例如见文献 [22-24]。最终，文献 [24] 以邦奇-戴维斯真空 [25] 作为初始条件，研究了宇宙早期德西特膨胀阶段 (当时还未被称为暴胀) 产生引力子的过程。

Although acknowledged as originating from vacuum fluctuation, the dynamics of primordial gravitational waves was first analysed classically as successive stages of parametric amplifications, either using a classical field and possibly fixing the initial conditions to match quantum vacuum fluctuations [21, 22, 24, 26, 27] or using mode functions [14, 28]. Another presentation, equivalent to the latter, consists in understanding the amplification of the waves as successive Bogoliubov transformations [29] where the initial state is chosen as the vacuum in an asymptotically Minkowski region. Finally, it was later recognised [30], moving to the Schrödinger picture, that the evolution puts the gravitational waves in a squeezed state. A good parallel presentation of the classical and quantum descriptions can be found in Ref. [31].

尽管原初引力波被公认为起源于真空涨落，但其动力学最初是通过分阶段参数放大进行经典分析的: 这类分析要么使用经典场，还可能固定初始条件来匹配量子真空涨落 [21, 22, 24, 26, 27]，要么使用模函数 [14, 28]。另一种表述与后者等价，它将引力波的放大理解为一系列博戈留波夫变换 [29]，其中初始态取渐近闵氏区域的真空态。后来，研究者在转向薛定谔绘景后认识到 [30]，演化会将引力波变为压缩态。经典描述和量子描述的平行阐述可参考文献 [31]。

In this section, we first proceed to the canonical quantisation of the field in the Heisenberg picture following [14]. This is the standard approach; we refer to Refs. [32-34] for textbooks dealing with scalar fields or gravitational waves. We then review different formal approaches to the evolution of a quantised gravitational wave field on an FLRW background. We begin by using a description in terms of a Bogoliubov transformation, then make the connection with mode functions and finally move to the Schrödinger picture, introducing squeezing parameters and the phase space representation of the state. We use these different approaches to discuss the mechanism of graviton creation in curved spacetime. This then leads to a discussion of how these particles back-react on the geometry. Finally, we use these analyses to compute the properties of primordial gravitational waves produced from the vacuum by the cosmological expansion and discuss their quantum origin. It should be noted that the exact same analyses on quantisation and time evolution can be repeated for scalar perturbations during inflation with the same type of equations [35].

在本节中，我们参照 [14]，首先对场进行海森堡绘景下的正则量子化。这是标准方法；关于标量场或引力波的相关内容可参考教材 [32-34]。随后我们回顾 FLRW 背景下量子化引力波场演化的不同形式方法。我们先从博戈留波夫变换的描述出发，之后建立它与模函数的联系，最后转向薛定谔绘景，引入压缩参数和量子态的相空间表示。我们利用这些不同方法讨论弯曲时空中引力子的产生机制，进而讨论这些粒子对几何的反作用，最后基于这些分析计算宇宙膨胀从真空产生的原初引力波的性质，并讨论它们的量子起源。需要注意的是，这套关于量子化和时间演化的分析可以完全照搬，用于研究暴胀时期的标量微扰，二者满足同一类方程 [35]。

Canonical Quantisation and Bogoliubov Transformation

正则量子化与博戈留波夫变换

Let us consider one of the two fields μ_λ in Eq. (21). It so happens that for the study of time evolution in terms of Bogoliubov transformations and squeezing, it is useful, and standard [30], to keep the total derivative that was dropped in the process of integration by part between Eqs. (20) and (21). The Lagrangian thus obtained reads

我们来考察式 (21) 中两个场之一 μ_λ。在用博戈留波夫变换和压缩研究时间演化时，保留式 (20) 和 (21) 之间分部积分过程中舍去的全导数是有用且符合常规做法 [30] 的，由此得到的拉格朗日量为

$$L_\lambda = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left[\left(\mu_\lambda'\right)^2 - 2\mathcal{H}\mu_\lambda'\mu_\lambda - \partial_i\mu_\lambda\partial^i\mu_\lambda + \mathcal{H}^2\mu_\lambda^2\right]. \tag{22}$$

The canonically conjugate momentum to μ_λ is

μ_λ 对应的正则共轭动量为

$$\pi_\lambda(\mathbf{x},\eta) = \frac{\delta L_\lambda}{\delta\mu_\lambda'} = \mu_\lambda' - \mathcal{H}\mu_\lambda, \tag{23}$$

so the Hamiltonian reads

因此哈密顿量为

$$H_\lambda = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left[\pi_\lambda^2 + \mathcal{H}\left(\pi_\lambda\mu_\lambda + \mu_\lambda\pi_\lambda\right) + \partial_i\mu_\lambda\partial^i\mu_\lambda\right], \tag{24}$$

the second term being written in a symmetric way, which is classically irrelevant but prepares for quantisation. We proceed to canonical quantisation by imposing equal-time canonical commutation relations (we now drop the λ subscripts)

第二项写为对称形式，这在经典层面无影响，但为量子化做好了准备。我们通过施加等时正则对易关系来进行正则量子化 (下文我们省略 λ 下标)

$$[\hat{\mu}(\mathbf{x},\eta),\hat{\pi}(\mathbf{x}',\eta)] = i\hbar\delta(\mathbf{x}-\mathbf{x}'), \tag{25a}$$

$$[\hat{\mu}(\mathbf{x},\eta),\hat{\mu}(\mathbf{x}',\eta)] = [\hat{\pi}(\mathbf{x},\eta),\hat{\pi}(\mathbf{x}',\eta)] = 0. \tag{25b}$$

Going to Fourier space, these relations are equivalent to

转换到傅里叶空间后，这些关系等价于

$$[\hat{\mu}_{\mathbf{k}}(\eta),\hat{\pi}_{\mathbf{k}'}(\eta)] = i\hbar\delta(\mathbf{k}+\mathbf{k}'), \tag{26a}$$

$$[\hat{\mu}_{\mathbf{k}}(\eta),\hat{\mu}_{\mathbf{k}'}(\eta)] = [\hat{\pi}_{\mathbf{k}}(\eta),\hat{\pi}_{\mathbf{k}'}(\eta)] = 0, \tag{26b}$$

and the Hamiltonian reads

此时哈密顿量为

$$\hat{H} = \int_{\mathbb{R}^{3+}} \mathrm{d}^3\mathbf{k}\hat{H}_{\pm\mathbf{k}} = \int_{\mathbb{R}^{3+}} \mathrm{d}^3\mathbf{k}\left[\hat{\pi}_{\mathbf{k}}\hat{\pi}_{-\mathbf{k}} + \mathcal{H}(\hat{\pi}_{\mathbf{k}}\hat{\mu}_{-\mathbf{k}} + \hat{\mu}_{\mathbf{k}}\hat{\pi}_{-\mathbf{k}}) + k^2\hat{\mu}_{\mathbf{k}}\hat{\mu}_{-\mathbf{k}}\right], \tag{27}$$

where $\hat{H}_{\pm\mathbf{k}}$ is the Hamiltonian for the $\pm\mathbf{k}$ sector. Observe that, as required by homogeneity, only the modes $\pm\mathbf{k}$ are coupled and the coupling only depends on the norm k , as required by isotropy. In order to expand $\hat{H}$ into a sum of independent Hamiltonians $\hat{H}_{\pm\mathbf{k}}$ for the bi-modes $\pm\mathbf{k}$, we restrict the integration to be over the top-half of the Fourier space, denoted by $\mathbb{R}^{3+}$, e.g., by selecting only the vectors $\mathbf{k}$ with positive k_z component and dropping the original global factor of a half.

其中 $\hat{H}_{\pm\mathbf{k}}$ 是 $\pm\mathbf{k}$ 分支的哈密顿量。可见，正如均匀性要求的那样，只有 $\pm\mathbf{k}$ 模相互耦合，且耦合仅由范数 k 决定，这也满足各向同性的要求。为了将 $\hat{H}$ 展开为双模态 $\pm\mathbf{k}$ 对应的独立哈密顿量 $\hat{H}_{\pm\mathbf{k}}$ 之和，我们将积分范围限制在傅里叶空间的上半部分，记为 $\mathbb{R}^{3+}$ ，具体操作是只选取满足 k_z 分量为正的矢量 $\mathbf{k}$ ，并舍去原先整体的 1/2 因子。

Let us first analyse the evolution of one such pair of modes $\pm\mathbf{k}$ in a situation where the term in $\mathcal{H}$ can be neglected with respect to the others, so that $\hat{\mu}$ is just a free scalar field in Minkowski spacetime. With $\mathcal{H} \to 0$, the Hamiltonian $\hat{H}$ is time-independent, and we can introduce the usual creation/annihilation operators for a real scalar field

我们首先分析当 $\mathcal{H}$ 项相对于其他项可以忽略，$\hat{\mu}$ 退化为闵可夫斯基时空中的自由标量场时，一对这样的模 $\pm\mathbf{k}$ 的演化。在 $\mathcal{H} \to 0$ 条件下，哈密顿量 $\hat{H}$ 不依赖于时间，我们可以引入实标量场常用的产生/湮灭算符

$$\hat{\mu}_{\mathbf{k}}(\eta) = \sqrt{\frac{\hbar}{2k}}\left[\hat{a}_{\mathbf{k}}(\eta) + \hat{a}^{\dagger}_{-\mathbf{k}}(\eta)\right], \tag{28a}$$

$$\hat{\pi}_{\mathbf{k}}(\eta) = -i\sqrt{\frac{\hbar k}{2}}\left[\hat{a}_{\mathbf{k}}(\eta) - \hat{a}^{\dagger}_{-\mathbf{k}}(\eta)\right]. \tag{28b}$$

The equal-time commutation relations assume the standard form

等时对易关系取标准形式

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}\right] = \delta(\mathbf{k} - \mathbf{k}') \quad \text{and} \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = \left[\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}\right] = 0. \tag{29}$$

The Hamiltonian $\hat{H}_{\pm k}$ then separates into two harmonic oscillators of frequency k

此时哈密顿量 $\hat{H}_{\pm k}$ 分解为两个频率为 k 的简谐振子

$$\hat{H}^{(0)}_{\pm\mathbf{k}} = \hbar k\left(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}\right) + \hbar k\left(\hat{a}^{\dagger}_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} + \frac{1}{2}\right), \tag{30}$$

and the Heisenberg equations of motions

而海森堡运动方程为

$$i\hbar\frac{\mathrm{d}\hat{a}^{(\dagger)}_{\pm\mathbf{k}}}{\mathrm{d}\eta} = \left[\hat{a}^{(\dagger)}_{\pm\mathbf{k}}, \hat{H}^{(0)}_{\pm\mathbf{k}}\right]$$

give $\hat{a}_{\mathbf{k}}(\eta) = \hat{a}_{\mathbf{k}}(0)e^{-ik\eta}$. Including the friction term proportional to the Hubble function $\mathcal{H}$, the Hamiltonian now reads

给出 $\hat{a}_{\mathbf{k}}(\eta) = \hat{a}_{\mathbf{k}}(0)e^{-ik\eta}$ 。若计入与哈勃函数 $\mathcal{H}$ 成正比的摩擦项，哈密顿量变为

$$\hat{H}_{\pm\mathbf{k}} = \hbar k\left(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}\right) + \hbar k\left(\hat{a}^{\dagger}_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} + \frac{1}{2}\right) - i\mathcal{H}\hbar\left(\hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{-\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\right). \tag{31}$$

The additional term corresponds to an interaction with a time-dependent classical source, the expanding background, acting through $\mathcal{H}$. It couples the $\pm\mathbf{k}$ modes by creating/destroying pairs of particles with opposite momentum; $\hat{a}_{\mathbf{k}}$ is paired with $\hat{a}_{-\mathbf{k}}$ and similarly for their Hermitian conjugate. These terms are the only quadratic interaction terms that respect homogeneity. The Heisenberg equations of motions accordingly only mix $\hat{a}_{\mathbf{k}}$ with $\hat{a}^{\dagger}_{\mathbf{k}}$, namely

附加项对应于与通过 $\mathcal{H}$ 作用的含时经典源——膨胀背景——的相互作用。它通过产生/湮灭一对动量相反的粒子耦合 $\pm\mathbf{k}$ 模；$\hat{a}_{\mathbf{k}}$ 与 $\hat{a}_{-\mathbf{k}}$ 配对，它们的厄米共轭也同理。这些项是满足均匀性要求仅有的二次相互作用项。因此海森堡运动方程仅混合 $\hat{a}_{\mathbf{k}}$ 和 $\hat{a}^{\dagger}_{\mathbf{k}}$ ，即

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{pmatrix}\hat{a}_{\mathbf{k}}\\ \hat{a}^{\dagger}_{-\mathbf{k}}\end{pmatrix} = \begin{pmatrix}-ik\mathcal{H}\\ /k\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{k}}\\ \hat{a}^{\dagger}_{-\mathbf{k}}\end{pmatrix}. \tag{32}$$

The operators at any further time η can then be expressed as a linear combination of operators at an earlier time η_{in}

任意较晚时刻 η 的算符都可以表示为较早时刻 η_{in} 算符的线性组合

$$\begin{bmatrix} \hat{a}_{\mathbf{k}}(\eta) \\ \hat{a}^{\dagger}_{-\mathbf{k}}(\eta) \end{bmatrix} = \begin{bmatrix} \alpha_k(\eta) & \beta_k(\eta) \\ \beta^*_k(\eta) & \alpha^*_k(\eta) \end{bmatrix} \begin{bmatrix} \hat{a}_{\mathbf{k}}(\eta_{\text{ in }}) \\ \hat{a}^{\dagger}_{-\mathbf{k}}(\eta_{\text{ in }}) \end{bmatrix}. \tag{33}$$

The system (32) is equivalent to

方程组 (32) 等价于

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{pmatrix} \alpha_k \\ \beta^\star_k \end{pmatrix} = \begin{pmatrix} -ik\mathcal{H} \\ k \\ ik \end{pmatrix}\begin{pmatrix} \alpha_k \\ \beta^\star_k \end{pmatrix}, \tag{34}$$

with $\alpha_k(\eta_{\text{ in }}) = 1$ and $\beta_k(\eta_{\text{ in }}) = 0$ as initial conditions. One can check that Eq. (34) implies the quantity $|\alpha_k|^2 - |\beta_k|^2$ is conserved, while the commutation relations (29) impose

以 $\alpha_k(\eta_{\text{ in }}) = 1$ 和 $\beta_k(\eta_{\text{ in }}) = 0$ 作为初始条件。可以验证，式 (34) 表明量 $|\alpha_k|^2 - |\beta_k|^2$ 是守恒量，而对易关系 (29) 要求

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{35}$$

At any fixed η , a transformation like (33) respecting the condition (35) is called a Bogoliubov transformation [36]. Notice that the equations of motion, and so the Bogoliubov coefficients, only depend on the norm k . The evolution of the quantum field has thus been reduced to finding the coefficients of a time-dependent Bogoliubov transformation. A convenient way to analyse this situation is to introduce mode functions.

在任意固定 η 下，满足条件 (35) 的形如 (33) 的变换称为博戈留博夫变换 [36]。注意运动方程以及博戈留博夫系数仅依赖于范数 k 。因此量子场的演化被简化为求解含时博戈留博夫变换的系数。分析该情形的一种简便方法是引入模函数。

Mode Functions

模函数

Having observed that the dynamics only mixes $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{-\mathbf{k}}$, we have a basis on which to expand $\hat{\mu}$. Inserting Eq. (33) in the Fourier expansion of the field $\hat{\mu}$, we get

我们已经发现动力学仅混合 $\hat{a}_{\mathbf{k}}$ 和 $\hat{a}^{\dagger}_{-\mathbf{k}}$ ，因此得到了一个可用于展开 $\hat{\mu}$ 的基底。将式 (33) 代入场 $\hat{\mu}$ 的傅里叶展开中，可得

$$\hat{\mu}_{\mathbf{k}}(\eta) = u_k(\eta)\,\hat{a}_{\mathbf{k}}(\eta_{\text{in}}) + u^\star_k(\eta)\,\hat{a}^{\dagger}_{-\mathbf{k}}(\eta_{\text{in}}), \tag{36a}$$

$$\hat{\pi}_{\mathbf{k}}(\eta) = U_k(\eta)\,\hat{a}_{\mathbf{k}}(\eta_{\text{in}}) + U^\star_k(\eta)\,\hat{a}^{\dagger}_{-\mathbf{k}}(\eta_{\text{in}}), \tag{36b}$$

where u_k and U_k are defined by

其中 u_k 和 U_k 由下式定义

$$u_k(\eta) = \sqrt{\frac{\hbar}{2k}} \left[\alpha_k(\eta) + \beta_k^\star(\eta)\right], \tag{37a}$$

$$U_k(\eta) = -i\sqrt{\frac{\hbar k}{2}} \left[\alpha_k(\eta) - \beta_k^\star(\eta)\right]. \tag{37b}$$

Using these functions, we get the so-called mode expansion of the field $\widehat{\mu}$

利用这些函数，我们可以得到场 $\widehat{\mu}$ 的所谓模展开

$$\widehat{\mu}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}.\mathbf{x}} u_k(\eta)\,\hat{a}_{\mathbf{k}}(\eta_{\text{in}}) + e^{-i\mathbf{k}.\mathbf{x}} u_k^\star(\eta)\,\hat{a}_{\mathbf{k}}^\dagger(\eta_{\text{in}})\right], \tag{38}$$

and a similar expression for $\widehat{\pi}$ with U_k instead of u_k . It can be checked from Eq. (33) that u_k simply obeys the same equation of motion (4) as the classical field $\mu(\mathbf{k},\eta)$; the momentum mode function U_k is then determined by

用 U_k 代替 u_k 即可得到 $\widehat{\pi}$ 的类似表达式。可以从式 (33) 验证，u_k 与经典场 $\mu(\mathbf{k},\eta)$ 一样满足相同的运动方程 (4)；动量模函数 U_k 由下式确定

$$u_k' = \mathcal{H} u_k + U_k. \tag{39}$$

Finally, the conserved quantity $|\alpha_k|^2 - |\beta_k|^2$ maps to the Wronskian $W\left(u_k, u_k^\star\right) = u_k^\star u_k' - u_k^{\star\prime} u_k$, which is a conserved quantity of Eq. (4), so the condition (35) translates in the normalisation

最后，守恒量 $|\alpha_k|^2 - |\beta_k|^2$ 对应朗斯基行列式 $W\left(u_k, u_k^\star\right) = u_k^\star u_k' - u_k^{\star\prime} u_k$ ，它是式 (4) 的守恒量，因此条件 (35) 转化为归一化条件

$$W\left(u_k, u_k^\star\right) = -i\hbar. \tag{40}$$

Any function u_k solution of Eq. (4) and which satisfies the normalisation condition of the Wronskian is called a mode function.

任何满足式 (4) 且满足朗斯基行列式归一化条件的函数 u_k 都称为模函数。

We now have a dictionary between the Bogoliubov and mode function presentations. Solving the system Eq. (34) with initial conditions $\alpha_k(\eta_{\text{in}}) = 1$ and $\beta_k(\eta_{\text{in}}) = 0$ is equivalent to solving Eq. (4) for u_k with initial conditions $u_k(\eta_{\text{in}}) = \sqrt{\hbar/2k}$ and $u_k'(\eta_{\text{in}}) = -i\sqrt{\hbar k/2} + \mathcal{H}(\eta_{\text{in}})$, U_k being determined by Eq. (39). Using mode functions the quantum dynamics reduces to the classical one. This justifies the classical treatment used in works cited in the introduction of this section; it is simply a consequence of working at linear order, and we will encounter other manifestations of this fact when studying phase space representation.

现在我们已经得到了博戈留波夫表示与模函数表示之间的对应关系。求解满足初始条件 $\alpha_k(\eta_{\text{in}}) = 1$ 和 $\beta_k(\eta_{\text{in}}) = =0$ 的方程组 (34)，等价于求解满足初始条件 $u_k(\eta_{\text{in}}) = \sqrt{\hbar/2k}$ 和 $u'_k(\eta_{\text{in}}) = -i\sqrt{\hbar k/2} + \mathcal{H}(\eta_{\text{in}}), U_k$ (由式 (39) 确定) 的 u_k 对应的式 (4)。利用模函数可将量子动力学约化为经典动力学。这就证明了本节引言所引工作中采用经典处理的合理性；这只是线性阶处理的直接结论，我们在研究相空间表示时还会看到这一结论的其他体现。

Squeezed States

压缩态

The time evolution was described so far in the Heisenberg picture. We now show how to move to the Schrödinger picture and introduce the squeezing formalism. This formulation was initially proposed in Ref. [30], and we use conventions matching those of Ref. [37]. Without loss of generality, the Bogoliubov coefficients (33) can be parametrised using three real coefficients r_k, φ_k and θ_k through

到目前为止，我们都是在海森堡绘景下描述时间演化。现在我们说明如何转换到薛定谔绘景，并介绍压缩形式体系。该表述最初由文献 [30] 提出，我们采用的约定与文献 [37] 一致。不失一般性，博戈留波夫系数 (33) 可通过三个实系数 r_k, φ_k 和 θ_k 参数化为

$$\alpha_k(\eta) = e^{-i\theta_k(\eta)} \cosh\left[r_k(\eta)\right], \tag{41a}$$

$$\beta_k(\eta) = -e^{i[\theta_k(\eta)+2\varphi_k(\eta)]} \sinh\left[r_k(\eta)\right], \tag{41b}$$

where r_k and φ_k are, respectively, called the squeezing parameter and angle, collectively referred to as the squeezing parameters. We define the two-mode squeezing and the two-mode rotation operators by

其中 r_k 和 φ_k 分别被称为压缩参数和压缩角，统称为压缩参数。我们由此定义双模压缩算符和双模旋转算符:

$$\hat{S}(r_k, \varphi_k) = \exp\left[\int_{\mathbb{R}^{3+}} \mathrm{d}^3\mathbf{k} \left(r_k e^{-2i\varphi_k} \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} - r_k e^{2i\varphi_k} \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}}\right)\right], \tag{42a}$$

$$\widehat{R}(\theta_k) = \exp\left[-i\int_{\mathbb{R}^{3+}} \mathrm{d}^3\mathbf{k}\theta_k \left(\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}\right)\right], \tag{42b}$$

in which the integrals are again, as in Eq. (27), performed over half the Fourier space and the creation and annihilation operators are understood to be evaluated at η_{in} . The operators $\hat{S}$ and $\widehat{R}$ defined through Eq. (42) are unitary, and one can check that

与式 (27) 一样，积分仅在半个傅里叶空间上进行，产生湮灭算符被理解为在 η_{in} 处取值。由式 (42) 定义的算符 $\hat{S}$ 和 $\widehat{R}$ 是幺正算符，可以验证

$$\hat{a}^{(\dagger)}_{\pm\mathbf{k}}(\eta) = \widehat{R}^\dagger(\theta_k) \hat{S}^\dagger(r_k, \varphi_k) \hat{a}^{(\dagger)}_{\pm\mathbf{k}}(\eta_{\text{in}}) \hat{S}(r_k, \varphi_k) \widehat{R}(\theta_k), \tag{43}$$

where the parameters are that of Eq. (41) and we have made their time dependence implicit for display convenience. The time evolution equation (33) is seen to correspond to the application of a rotation of parameter $\theta_k(\eta)$ followed by a squeezing of parameters $r_k(\eta)$ and $\varphi_k(\eta)$ on the operators.

式中参数取自式 (41)，为方便展示，我们省略了其时间依赖关系。可见时间演化方程 (33) 对应于: 先对算符做参数为 $\theta_k(\eta)$ 的旋转，再做参数为 $r_k(\eta)$ 和 $\varphi_k(\eta)$ 的压缩。

Any operator $\widehat{O}(\eta)$ in the Heisenberg picture can be written as a combination of $\hat{a}_{\pm\mathbf{k}}^{(\dagger)}(\eta)$, so we have

海森堡绘景中的任意算符 $\widehat{O}(\eta)$ 都可以表示为 $\hat{a}_{\pm\mathbf{k}}^{(\dagger)}(\eta)$ 的组合，因此我们有

$$\left\langle\Psi(\eta_{\rm in})\left|\widehat{O}(\eta)\right|\Psi(\eta_{\rm in})\right\rangle=\left\langle\Psi(\eta_{\rm in})\left|\widehat{R}^\dagger\hat{S}^\dagger\widehat{O}(\eta_{\rm in})\hat{S}\widehat{R}\right|\Psi(\eta_{\rm in})\right\rangle, \tag{44}$$

$$=\langle\Psi(\eta)\left|\widehat{O}(\eta_{\rm in})\right|\Psi(\eta)\rangle.$$

where $\left|\Psi(\eta)\right\rangle=\hat{S}\widehat{R}\left|\Psi(\eta_{\rm in})\right\rangle$ is the Schrödinger evolved state of the system. Choosing the waves to be initially in the vacuum of $\hat{a}_{\pm\mathbf{k}}^{(\dagger)}(\eta_{\rm in})$ for all modes $\mathbf{k}$ (we return to this point later) yields

其中 $\left|\Psi(\eta)\right\rangle=\hat{S}\widehat{R}\left|\Psi(\eta_{\rm in})\right\rangle$ 是系统按薛定谔绘景演化的态。取所有模 $\mathbf{k}$ 初始都处于 $\hat{a}_{\pm\mathbf{k}}^{(\dagger)}(\eta_{\rm in})$ 的真空态 (我们后文中会回到这一点)，可得

$$|\Psi(\eta)\rangle=\prod_{\mathbb{R}^{3+}}\hat{S}(r_k,\varphi_k)\widehat{R}(\theta_k)\left|0_{\mathbf{k}},0_{-\mathbf{k}}\right\rangle=\prod_{\mathbb{R}^{3+}}\left|2\mathrm{MS},r_k,\varphi_k\right\rangle, \tag{45}$$

where we have defined the two-mode squeezed state (2MS) for the modes $\pm\mathbf{k}$

其中我们已经为模 $\pm\mathbf{k}$ 定义了双模压缩态 (2MS)

$$\left|2\mathrm{MS},r_k,\varphi_k\right\rangle=\hat{S}(r_k,\varphi_k)\left|0_{\mathbf{k}},0_{-\mathbf{k}}\right\rangle=\frac{1}{\cosh(2r_{\mathbf{k}})}\sum_{n=0}^{+\infty}\left(-\tanh 2r_k e^{2i\varphi_k}\right)^n\left|n_{\mathbf{k}},n_{-\mathbf{k}}\right\rangle.$$

(46)

The last expression can be computed using a Baker-Campbell-Hausdorff formula on the squeezing operator, now restricted to a single $\pm\mathbf{k}$ sector [38], and $\left|n_{\mathbf{k}},n_{-\mathbf{k}}\right\rangle$ is the state with n particles in the mode $\mathbf{k}$ and $-\mathbf{k}$. Note that the rotation angle θ_k has dropped from Eq. (46) because the vacuum is invariant under the rotation operator and the product involved is over all directions.

利用压缩算符的贝克-坎贝尔-豪斯多夫公式 (限制在单个 $\pm\mathbf{k}$ 扇区 [38]) 可以计算最后这个表达式，且 $\left|n_{\mathbf{k}},n_{-\mathbf{k}}\right\rangle$ 是模 $\mathbf{k}$ 和 $-\mathbf{k}$ 含有 n 个粒子的态。注意旋转角 θ_k 已经从式 (46) 中消失，这是因为真空在旋转算符作用下不变，且乘积是对所有方向取值的。

Following Ref. [39], one can quickly derive the associated wavefunction of a single pair of modes by assuming that, at the initial time, the corresponding state is annihilated by both annihilation operators, i.e.,

按照文献 [39] 的方法，我们可以快速推导一对单模的关联波函数: 假设初始时刻对应的态被两个湮灭算符同时湮灭，即

$$\hat{a}_{\pm\mathbf{k}}(\eta_{\text{in}})|0_{\mathbf{k}},0_{-\mathbf{k}}\rangle = 0. \tag{47}$$

Since $\hat{S}$ is unitary $\left(\hat{S}^\dagger\hat{S} = \mathbb{1}\right)$, this is also

由于 $\hat{S}$ 是幺正算符 $\left(\hat{S}^\dagger\hat{S} = \mathbb{1}\right)$，上式也可写为

$$0 = \hat{S}(r_k,\varphi_k)\,\hat{a}_{\pm\mathbf{k}}\hat{S}^\dagger(r_k,\varphi_k)\,\hat{S}(r_k,\varphi_k)|0_{\mathbf{k}},0_{-\mathbf{k}}\rangle, \tag{48}$$

$$= \hat{S}(r_k,\varphi_k)\,\hat{a}_{\pm\mathbf{k}}\hat{S}^\dagger(r_k,\varphi_k)|2\text{MS},r_k,\varphi_k\rangle,$$

where the transformation on the left corresponds to the inverse of Eq. (43) for $\theta_k = 0$. Inverting the Bogoliubov transformation (33) and using Eq. (28), the relation (48) becomes

其中左侧的变换对应 $\theta_k ==0$ 时式 (43) 的逆变换。对博戈留波夫变换 (33) 求逆并代入式 (28)，关系 (48) 变为

$$\left[\hat{\mu}_{\pm\mathbf{k}} + \frac{i}{k}\left(\frac{1-i\gamma_{12}}{\gamma_{11}}\right)^{-1}\hat{\pi}_{\pm\mathbf{k}}\right]|2\text{MS},r_k,\varphi_k\rangle = 0, \tag{49}$$

where, anticipating the next section, we have introduced the matrix entries

为给下一节做铺垫，我们在这里引入矩阵元

$$\gamma_{11} = \cosh(2r_k) - \cos(2\varphi_k)\sinh(2r_k), \tag{50a}$$

$$\gamma_{12} = -\sin(2\varphi_k)\sinh(2r_k). \tag{50b}$$

Projecting Eq. (49) onto the $\mu_{\pm\mathbf{k}}$ -representation of the wavefunction by setting $\hat{\mu}_{\pm\mathbf{k}} \to \mu_{\pm\mathbf{k}}$ and $\hat{\pi}_{\pm\mathbf{k}} \to -i\hbar\partial/\partial\mu_{\mp\mathbf{k}}$, the wavefunction solution of Eq. (49) reads

通过设置 $\hat{\mu}_{\pm\mathbf{k}} \to \mu_{\pm\mathbf{k}}$ 和 $\hat{\pi}_{\pm\mathbf{k}} \to -i\hbar\partial/\partial\mu_{\mp\mathbf{k}}$，将式 (49) 投影到波函数的 $\mu_{\pm\mathbf{k}}$ 表象，可得式 (49) 的波函数解为

$$\Psi(\mu_{\mathbf{k}},\mu_{-\mathbf{k}}) = \sqrt{\frac{k}{\pi\hbar\gamma_{11}}}\,e^{-\frac{k}{\hbar}\frac{(1-i\gamma_{12})}{\gamma_{11}}\mu_{\mathbf{k}}\mu_{-\mathbf{k}}}, \tag{51}$$

which we normalised, using Eq. (16), to $\int|\Psi|^2\,\mathrm{d}\mu_{\mathbf{k}}\mathrm{d}\mu_{-\mathbf{k}} = 1$. Note that formally, the wavefunction is the projection of the relevant state on the basis $|\{\mu_{\pm\mathbf{k}}\}\rangle$, i.e., $\Psi(\mu_{\mathbf{k}},\mu_{-\mathbf{k}}) = \langle\mu_{\pm\mathbf{k}} \mid 2\text{MS},r_k,\varphi_k\rangle$.

我们利用式 (16) 将其归一化为 $\int|\Psi|^2\,\mathrm{d}\mu_{\mathbf{k}}\mathrm{d}\mu_{-\mathbf{k}} = 1$。注意，形式上波函数是相关态在基底 $|\{\mu_{\pm\mathbf{k}}\}\rangle$ 上的投影，即 $\Psi(\mu_{\mathbf{k}},\mu_{-\mathbf{k}}) = \langle\mu_{\pm\mathbf{k}} \mid 2\text{MS},r_k,\varphi_k\rangle$.

When the squeezing parameters are those determined by Eqs. (41), this gives the wavefunction of any $\pm\mathbf{k}$ mode of the gravitational waves. One can also provide a description in terms of the squeezed state parameters only by recasting Eq. (33) into a set of differential equations involving only r_k, φ_k and θ_k . One finds that the system

当压缩参数由式 (41) 确定时，我们由此可得到引力波任意 $\pm\mathbf{k}$ 模式的波函数。我们也可以仅用压缩参数来描述: 将式 (33) 改写为一组仅包含 r_k, φ_k 和 θ_k 的微分方程，可发现该体系

$$\frac{\mathrm{d}r_k}{\mathrm{d}\eta} = -\mathcal{H}\cos\left(2\varphi_k\right), \tag{52a}$$

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}\eta} = -k + \mathcal{H}\coth\left(2r_k\right)\sin\left(2\varphi_k\right), \tag{52b}$$

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}\eta} = k - \mathcal{H}\tanh\left(r_k\right)\sin\left(2\varphi_k\right), \tag{52c}$$

should indeed hold. Note that the equations describing the time evolution of the squeezing parameters r_k and φ_k , namely, Eqs. (52a) and (52b), are independent of θ_k . These equations are however rarely solved directly, as it is easier to first solve Eq. (4) for the mode function, then deduce the Bogoliubov coefficients by inverting Eq. (37) and finally, using Eq. (41), obtain the expression of the squeezing parameters. The virtue of the squeezing formalism is rather to give a clear phase space representation of the system's evolution. Such representation can be obtained using the Wigner quasi-probability distribution [40] to which we now turn.

确实成立。注意，描述压缩参数 r_k 和 φ_k 时间演化的方程即式 (52a) 和 (52b) 与 θ_k 无关。不过这些方程很少直接求解，因为先求解模式函数的式 (4)，再通过反转式 (37) 推导出 Bogoliubov 系数，最后利用式 (41) 得到压缩参数的表达式会更简便。压缩形式体系的优点在于它能给出系统演化清晰的相空间表示。这种表示可以通过 Wigner 准概率分布 [40] 得到，我们现在就来讨论它。

Wigner Function

维格纳函数

Consider a system described by a density matrix $\hat{\rho}$ and represented by n -pairs of canonically conjugate hermitian operators $\widehat{X} = \{(\hat{q}_i, \hat{p}_i)\}_{i\in[1,n]}$ of the same dimension. The Wigner function is a function of $2n$ phase space variables $X = \{(q_i, p_i)\}_{i\in[1,n]}$ defined by

考虑一个由密度矩阵 $\hat{\rho}$ 描述的系统，该系统由 n 对维数相同的正则共轭厄米算符 $\widehat{X} = \{(\hat{q}_i, \hat{p}_i)\}_{i\in[1,n]}$ 表示。维格纳函数是由如下方式定义的、关于 $2n$ 个相空间变量 $X = \{(q_i, p_i)\}_{i\in[1,n]}$ 的函数

$$W(X) = \frac{1}{(2\pi\hbar)^n}\int \mathrm{d}^n\vec{x}e^{-i\vec{p}.\vec{x}/\hbar}\left\langle \vec{q} + \frac{\vec{x}}{2}\left|\hat{\rho}\right|\vec{q} - \frac{\vec{x}}{2}\right\rangle, \tag{53}$$

where the states entering the averaging are product eigenstates of $\hat{q}_i$. The righthand side of Eq. (53) is the Weyl transform of $\hat{\rho}/(2\pi)^n$. This transform maps any observable $\widehat{O}$, which is a function of operators in $\widehat{X}$, to a function $\widetilde{O}(X)$ of the associated classical variables X . A crucial property is that the expectation value of any such observable $\widehat{O}$ can be computed by treating the Wigner function as a probability measure for the Weyl transform

其中参与求平均的态是 $\hat{q}_i$ 的乘积本征态。式 (53) 的右侧是 $\hat{\rho}/(2\pi)^n$ 的外尔变换。该变换将任意可观测量 $\hat{O}$ (它是 $\hat{X}$ 中算符的函数) 映射为对应经典变量 X 的函数 $\widetilde{O}(X)$ 。一个关键性质是，只要将维格纳函数视作外尔变换的概率测度，就可以计算任意这类可观测量 $\hat{O}$ 的期望值

$$\langle\hat{O}\rangle = \mathbb{E}\left[\widetilde{O}(X)\right] = \int W(X)\,\widetilde{O}(X)\,\mathcal{D}X, \tag{54}$$

where the integral is over all the relevant variables in X and we denoted $\mathbb{E}$ the stochastic average with respect to the Wigner function. Equation (54) then allows to compute averages using the Wigner function as any classical phase space probability distribution. Finally, the von Neumann equation of motion for the density matrix can be mapped into an equation of motion for the Wigner function, namely [41],

其中积分覆盖 X 中所有相关变量，我们用 $\mathbb{E}$ 表示对维格纳函数的随机平均。式 (54) 表明，我们可以像使用任意经典相空间概率分布一样，用维格纳函数计算平均值。最后，密度矩阵的冯·诺依曼运动方程可以映射为维格纳函数的运动方程，即 [41]:

$$i\hbar\dot{W}(X) = H(\vec{q},\vec{p}) \star W - W \star H(\vec{q},\vec{p}), \tag{55}$$

where the non-commutative $\star$ -product is defined by

其中非对易 $\star$ 积定义为

$$f(\vec{q},\vec{p}) \star g(\vec{q},\vec{p}) = f\left(\vec{q} + \frac{i\hbar}{2}\partial_{\vec{p}}, \vec{p} - \frac{i\hbar}{2}\partial_{\vec{q}}\right) g(\vec{q},\vec{p}), \tag{56a}$$

$$= f(\vec{q},\vec{p})\, g\left(\vec{q} - \frac{i\hbar}{2}\partial_{\vec{p}}, \vec{p} + \frac{i\hbar}{2}\partial_{\vec{q}}\right). \tag{56b}$$

The Wigner function therefore furnishes a complete representation of the state of the system and its evolution in phase space.

因此，维格纳函数提供了系统状态及其在相空间中演化的完整表示。

Two remarks are in order here. First, in general, the Wigner function is not everywhere positive making it only a quasi-probability distribution. It can be shown that, for pure states, it is everywhere positive only when it takes the form [42]

这里有两点需要说明。第一，一般情况下维格纳函数并非处处为正，因此它只是一个准概率分布。可以证明，对于纯态，仅当它取如下形式时维格纳函数才处处为正 [42]

$$W(X) = \frac{1}{(\pi\hbar)^n\sqrt{\det\gamma}} \exp\left(-\frac{X^{\mathrm{T}}\gamma^{-1}X}{\hbar}\right), \tag{57}$$

which is completely determined by γ , the covariance matrix, defined by

它完全由协变矩阵 γ 确定，协变矩阵定义为

$$\gamma_{ab} = \left\langle \widehat{X}_a \widehat{X}_b + \widehat{X}_a \widehat{X}_b \right\rangle . \tag{58}$$

Such states are called Gaussian states and are widely used in quantum optics; see Ref. [43] for a review. Second, for evolution under a quadratic Hamiltonian $H\left(\widehat{X}\right)$, the dynamics (55) simply reduces to the classical Liouville equation

这类态被称为高斯态，在量子光学中被广泛使用，综述可见文献 [43]。第二，当系统在二次哈密顿量 $H\left(\widehat{X}\right)$ 下演化时，动力学方程 (55) 可直接约化为经典刘维尔方程

$$\dot{W}(X) = \{H(X), W(X)\}, \tag{59}$$

the curly brackets denoting the usual classical Poisson brackets. A detailed derivation in the special case of cosmological perturbations can be found in Appendix H of Ref. [37].

花括号表示通常的经典泊松括号。关于宇宙学扰动特殊情形的详细推导可见文献 [37] 的附录 H。

Equation (59) can be solved by the method of characteristics, i.e., by evolving the initial distribution along the classical trajectories given by H. This is another manifestation of the fact that, at quadratic order, the quantum dynamics reduces to the classical one. In addition, this implies that an initially Gaussian state will remain Gaussian under a quadratic Hamiltonian and that its evolution is thus summarised in that of its covariance matrix γ.

式 (59) 可以用特征线法求解，也就是让初始分布沿 H 给出的经典轨迹演化。这再次证明，在二次阶下量子动力学可以约化为经典动力学。此外这也说明，初始的高斯态在二次哈密顿量下演化后仍保持高斯性，因此其演化可以完全由其协变矩阵 γ 的演化概括。

Both of the above discussed simplifications apply to cosmological perturbations at linear order, to which we return by considering a pair of modes $\pm\mathbf{k}$. These two degrees of freedom represented by the four operators $\widehat{\mu}_{\pm\mathbf{k}}$ and $\widehat{\pi}_{\pm\mathbf{k}}$. These four operators are not hermitian and related to one another by hermitian conjugation. We can however build two such pairs of operators by taking the real and imaginary parts of $\widehat{\mu}_{\pm k}$ and $\widehat{\pi}_{\pm k}$ up to a factor of $\sqrt{2}$, introduced for further convenience, namely

上述两个简化性质都适用于线性阶的宇宙学扰动，我们接下来讨论一对模 $\pm\mathbf{k}$。这两个自由度由四个算符 $\widehat{\mu}_{\pm\mathbf{k}}$ 和 $\widehat{\pi}_{\pm\mathbf{k}}$ 表示。这四个算符不是厄米算符，彼此之间满足厄米共轭关系。不过我们可以通过取 $\widehat{\mu}_{\pm k}$ 和 $\widehat{\pi}_{\pm k}$ 的实部与虚部，再乘上一个为后续计算方便引入的因子 $\sqrt{2}$，构造出两对厄米正则共轭算符，即

$$\widehat{\mu}^{\mathrm{R}}_{\mathbf{k}} = \frac{\widehat{\mu}_{\mathbf{k}} + \widehat{\mu}^{\dagger}_{\mathbf{k}}}{\sqrt{2}}, \; \widehat{\mu}^{\mathrm{I}}_{\mathbf{k}} = \frac{\widehat{\mu}_{\mathbf{k}} - \widehat{\mu}^{\dagger}_{\mathbf{k}}}{\sqrt{2}i} \tag{60a}$$

$$\widehat{\pi}^{\mathrm{R}}_{\mathbf{k}} = \frac{\widehat{\pi}_{\mathbf{k}} + \widehat{\pi}^{\dagger}_{\mathbf{k}}}{\sqrt{2}}, \; \widehat{\pi}^{\mathrm{I}}_{\mathbf{k}} = \frac{\widehat{\pi}_{\mathbf{k}} - \widehat{\pi}^{\dagger}_{\mathbf{k}}}{\sqrt{2}i}. \tag{60b}$$

One can straightforwardly check that those are indeed hermitian and canonically conjugate, i.e., $\left[\widehat{\mu}^{\mathrm{S}}_{k}, \widehat{\pi}^{\mathrm{S}'}_{k'}\right] = i\hbar\delta(\mathbf{k} - \mathbf{k}')\,\delta_{\mathrm{S,S'}}$ and $\left[\widehat{\mu}^{\mathrm{S}}_{k}, \widehat{\mu}^{\mathrm{S}'}_{k'}\right] = \left[\widehat{\pi}^{\mathrm{S}}_{k}, \widehat{\pi}^{\mathrm{S}'}_{k'}\right] = 0$ where $\mathrm{S} = \mathrm{R}, \mathrm{I}$. We arrange them into $\widehat{X}_{\mathrm{R/I}} = \left(k^{1/2}\widehat{\mu}^{\mathrm{R}}_{\pm\mathbf{k}}, k^{-1/2}\widehat{\pi}^{\mathrm{R}}_{-\mathbf{k}}, k^{1/2}\widehat{\mu}^{\mathrm{I}}_{-\mathbf{k}}, k^{-1/2}\widehat{\pi}\right.$

, vector in which we have introduced factors of k to give the same dimension to all entries; the associated vector of classical variables is denoted $X_{\mathrm{R/I}}$. The Wigner function with respect to these variables is defined by

> 我们可以直接验证这些算符确实是厄米的，且满足正则共轭关系，即 $\left[\hat{\mu}_{k}^{\mathrm{S}},\hat{\pi}_{k'}^{\mathrm{S'}}\right]=i\hbar\delta\left(\mathbf{k}-\mathbf{k}'\right)\delta_{\mathrm{S,S'}}$ 和 $\left[\hat{\mu}_{k}^{\mathrm{S}},\hat{\mu}_{k'}^{\mathrm{S'}}\right]=\left[\hat{\pi}_{k}^{\mathrm{S}},\hat{\pi}_{k'}^{\mathrm{S'}}\right]=0$，其中 S = R =I。我们将它们整理为 $\hat{X}_{\mathrm{R/I}}=\left(k^{1/2}\hat{\mu}_{\pm\mathbf{k}}^{\mathrm{R}},k^{-1/2}\hat{\pi}_{-\mathbf{k}}^{\mathrm{R}},k^{1/2}\hat{\mu}_{-\mathbf{k}}^{\mathrm{I}},k^{-1/2}\hat{\pi}_{-\mathbf{k}}^{\mathrm{R}}\right)$，在该向量中我们引入了 k 因子，使所有分量量纲一致；对应的经典变量向量记为 $X_{\mathrm{R/I}}$ 。关于这些变量的维格纳函数定义如下

$$W_{\pm\mathbf{k}}\left(X_{\pm\mathbf{k}}\right)=\frac{1}{\left(2\pi\hbar\right)^{2}}\int e^{-\frac{i}{\hbar}\left(\pi_{\mathbf{k}}^{\mathrm{R}}x+\pi_{\mathbf{k}}^{\mathrm{I}}y\right)}$$

$$\left\langle\mu_{\mathbf{k}}^{\mathrm{R}}+\frac{x}{2},\mu_{\mathbf{k}}^{\mathrm{I}}+\frac{y}{2}\left|\hat{\rho}_{\mathbf{k}}\right|\mu_{\mathbf{k}}^{\mathrm{R}}-\frac{x}{2},\mu_{\mathbf{k}}^{\mathrm{I}}-\frac{y}{2}\right\rangle\mathrm{d}x\,\mathrm{d}y. \tag{61}$$

In terms of the variables (60), the Hamiltonian $\hat{H}_{\pm k}$ separates into two equal Hamiltonian over the R/I sectors that thus evolve independently

> 在变量 (60) 下，哈密顿量 $\hat{H}_{\pm k}$ 可分解为 R/I 扇区上两个相等的哈密顿量，因此两个扇区独立演化

$$\begin{aligned}\hat{H}&=\frac{\hbar}{2}\int_{\mathbb{R}^{3+}}\mathrm{d}^{3}\mathbf{k}\sum_{\mathrm{s=R,I}}\left[\left(\hat{\pi}_{\mathbf{k}}^{\mathrm{S}}\right)^{2}+2\mathcal{H}\left(\hat{\mu}_{\mathbf{k}}^{\mathrm{S}}\hat{\pi}_{\mathbf{k}}^{\mathrm{S}}+\hat{\pi}_{\mathbf{k}}^{\mathrm{S}}\hat{\mu}_{\mathbf{k}}^{\mathrm{S}}\right)+k^{2}\left(\hat{\mu}_{\mathbf{k}}^{\mathrm{S}}\right)^{2}\right]\\&=\int_{\mathbb{R}^{3+}}\mathrm{d}^{3}\mathbf{k}\sum_{\mathrm{S=R,I}}\hat{H}_{\mathbf{k}}^{\mathrm{S}}.\end{aligned} \tag{62}$$

Similarly, the wavefunction (51) factorises into a product of two wavefunctions over each sector $\Psi\left(\mu_{k},\mu_{-k}\right)=\Psi\left(\mu_{k}^{\mathrm{R}}\right)\Psi\left(\mu_{k}^{\mathrm{I}}\right)$ with

> 同理，波函数 (51) 可分解为每个扇区上两个波函数的乘积 $\Psi\left(\mu_{k},\mu_{-k}\right)=\Psi\left(\mu_{k}^{\mathrm{R}}\right)\Psi\left(\mu_{k}^{\mathrm{I}}\right)$，满足

$$\Psi\left(\mu_{k}^{\mathrm{S}}\right)=\left(\frac{k}{\pi\hbar\gamma_{11}}\right)^{1/4}e^{-\frac{k}{2\hbar}\frac{\left(1-i\gamma_{12}\right)}{\gamma_{11}}\left(\mu_{k}^{\mathrm{S}}\right)^{2}}, \tag{63}$$

and the covariance matrix is block diagonal in the R/I partition $\gamma=\gamma^{\mathrm{R}}\bigoplus\gamma^{\mathrm{I}}$. These separations are in fact imposed by the homogeneity of the state that requires $\left\langle\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}^{\dagger}\right\rangle=\left\langle\hat{a}_{\mathbf{k}}^{2}\right\rangle=0$, which can be recast in the vanishing of all R/I cross terms [44]. Equation (63) is nothing else than the wavefunction of a one-mode squeezed state of parameter r_{k},φ_{k} [45]. Going from the $\pm\mathbf{k}$ operators to the R/I operators allows to view a two-mode squeezed state as a product of two one-mode squeezed states. This fact can be directly seen by factorizing the two-mode squeezing operator $\hat{S}\left(r_{k},\varphi_{k}\right)$ into to two one-mode squeezing operators for the R/I creation/annihilation operators defined via Eq. (28) where $\pm\mathbf{k}$ operators are replaced by R/I operators. Such transformations are studied in details in Ref. [46].

> 且协方差矩阵在 R/I 分块 $\gamma=\gamma^{\mathrm{R}}\bigoplus\gamma^{\mathrm{I}}$ 下是块对角的。这些分离实际上是由态的齐次性要求 $\left\langle\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}^{\dagger}\right\rangle=\left\langle\hat{a}_{\mathbf{k}}^{2}\right\rangle=0$ 所强加的，该条件可等价表述为所有 R/I 交叉项为零 [44]。式 (63) 正是参数为 r_{k},φ_{k} 的单模压缩态的波函数 [45]。从 $\pm\mathbf{k}$ 算符过渡到 R/I 算符后，我们可以将一个双模压缩态看作两个单模压缩态的乘积。将双模压缩算符 $\hat{S}\left(r_{k},\varphi_{k}\right)$ 因子分解为对应 R/I 产生/湮灭算符的两个单模压缩算符，就能直接看出这一性质，其中 R/I 产生/湮灭算符通过式 (28) 定义，仅将式中 $\pm\mathbf{k}$ 算符替换为 R/I 算符。这类变换的详细讨论可见文献 [46]。

Since the wavefunction (63) is Gaussian, then so is the associated Wigner function W^{S} ; vacuua and squeezed states are indeed Gaussian states. Note that their Gaussianity is preserved by the evolution because $\widehat{H}_k^{\mathrm{S}}$ is quadratic. The Wigner function (61) also factorises into $W_{\pm k} = W^{\mathrm{R}}\left(\widehat{\mu}_k^{\mathrm{R}}, \widehat{\pi}_k^{\mathrm{R}}\right) W^{\mathrm{I}}\left(\widehat{\mu}_k^{\mathrm{I}}, \widehat{\pi}_k^{\mathrm{I}}\right)$. Both sectors have identical covariance matrix, namely,

由于波函数 (63) 是高斯型的，因此对应的维格纳函数 W^{S} 也是高斯型的；真空态和压缩态确实都是高斯态。注意到由于 $\widehat{H}_k^{\mathrm{S}}$ 是二次型，演化过程会保持其高斯性。式 (61) 的维格纳函数还可以分解为 $W_{\pm k} = W^{\mathrm{R}}\left(\widehat{\mu}_k^{\mathrm{R}}, \widehat{\pi}_k^{\mathrm{R}}\right) W^{\mathrm{I}}\left(\widehat{\mu}_k^{\mathrm{I}}, \widehat{\pi}_k^{\mathrm{I}}\right)$ 。两个部分具有相同的协方差矩阵，即

$$\gamma^{\mathrm{S}} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \tag{64}$$

with

其中

$$\gamma_{11} = \frac{2k}{\hbar}\left\langle\left(\widehat{\mu}_{\mathbf{k}}^{\mathrm{R}}\right)^2\right\rangle = \frac{2k}{\hbar}\left\langle\left(\widehat{\mu}_{\mathbf{k}}^{\mathrm{I}}\right)^2\right\rangle = \frac{k}{\hbar}\left\langle\left\{\widehat{\mu}_{\mathbf{k}}, \widehat{\mu}_{\mathbf{k}}^{\dagger}\right\}\right\rangle, \tag{65a}$$

$$\gamma_{12} = \gamma_{21} = \frac{1}{\hbar}\left\langle\widehat{\mu}_{\mathbf{k}}^{\mathrm{R}}\widehat{\pi}_{\mathbf{k}}^{\mathrm{R}} + \widehat{\pi}_{\mathbf{k}}^{\mathrm{R}}\widehat{\mu}_{\mathbf{k}}^{\mathrm{R}}\right\rangle = \frac{1}{\hbar}\left\langle\widehat{\mu}_{\mathbf{k}}^{\mathrm{I}}\widehat{\pi}_{\mathbf{k}}^{\mathrm{I}} + \widehat{\pi}_{\mathbf{k}}^{\mathrm{I}}\widehat{\mu}_{\mathbf{k}}^{\mathrm{I}}\right\rangle = \frac{1}{\hbar}\left\langle\widehat{\mu}_{\mathbf{k}}\widehat{\pi}_{\mathbf{k}}^{\dagger} + \widehat{\pi}_{\mathbf{k}}\widehat{\mu}_{\mathbf{k}}^{\dagger}\right\rangle,$$

(65b)

$$\gamma_{22} = \frac{2}{\hbar k}\left\langle\left(\widehat{\pi}_{\mathbf{k}}^{\mathrm{R}}\right)^2\right\rangle = \frac{2}{\hbar k}\left\langle\left(\widehat{\pi}_{\mathbf{k}}^{\mathrm{I}}\right)^2\right\rangle = \frac{1}{\hbar k}\left\langle\left\{\widehat{\pi}_{\mathbf{k}}, \widehat{\pi}_{\mathbf{k}}^{\dagger}\right\}\right\rangle, \tag{65c}$$

where we defined the anti-commutator of two operators $\{\widehat{A}, \widehat{B}\} = \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$, and we expressed the entries of the covariance matrix in terms of two-point function of the original $\widehat{\mu}_k$ and $\widehat{\pi}_k$ operators (one can also check that $\left\langle\widehat{\mu}_k\widehat{\pi}_k + \widehat{\pi}_k^{\dagger}\widehat{\mu}_k^{\dagger}\right\rangle = 0$). Using Eq. (65) and the parametrisation (41), the covariance matrix can be conveniently expressed in terms of the squeezing parameters

此处我们定义了两个算符 $\{\widehat{A}, \widehat{B}\} = \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$ 的反对易子，并且将协方差矩阵的元表示为原始 $\widehat{\mu}_k$ 算符和 $\widehat{\pi}_k$ 算符的两点关联函数 (也可以验证 $\left\langle\widehat{\mu}_k\widehat{\pi}_k + \widehat{\pi}_k^{\dagger}\widehat{\mu}_k^{\dagger}\right\rangle = 0$ 成立)。利用式 (65) 和参数化 (41)，协方差矩阵可以方便地用压缩参数表示为

$$\gamma_{11} = \cosh(2r_k) - \cos(2\varphi_k)\sinh(2r_k), \tag{66a}$$

$$\gamma_{12} = \gamma_{21} = -\sin(2\varphi_k)\sinh(2r_k), \tag{66b}$$

$$\gamma_{22} = \cosh(2r_k) + \cos(2\varphi_k)\sinh(2r_k), \tag{66c}$$

where the expressions for γ_{11} and γ_{12} correspond to those defined earlier when computing the wavefunction. Finally, in order visualize this probability distribution, we compute its contour levels. Owing to Gaussianity, those are ellipses whose parameters can be computed through diagonalizing the quadratic form appearing in the argument of the exponential in Eq. (57). It is readily done by performing a rotation in phase space $\widetilde{X}^{\mathrm{S}} = R(-\varphi_k) X^{\mathrm{S}}$ so that the covariance matrix of X^{S} reads

其中 γ_{11} 和 γ_{12} 的表达式就是我们之前计算波函数时定义的形式。最后，为了可视化这个概率分布，我们计算它的等高线。由于其高斯性，这些等高线都是椭圆，椭圆参数可以通过对角化式 (57) 指数宗量中出现的二次型得到。这可以通过在相空间中做旋转 $\widetilde{X}^{\mathrm{S}} = R(-\varphi_k) X^{\mathrm{S}}$ 轻松完成，旋转后 X^{S} 的协方差矩阵变为

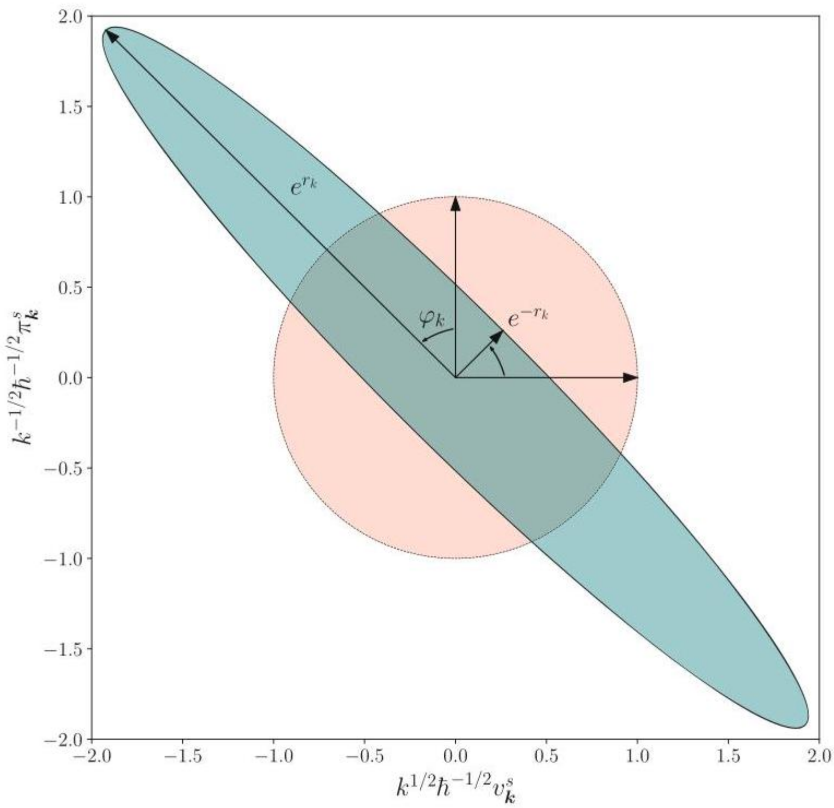
$$\left(\widetilde{\gamma}^{\mathrm{S}}\right)^{-1} = \begin{pmatrix} e^{2r_k} & 0 \\ 0 & e^{-2r_k} \end{pmatrix}. \tag{67}$$

Some contour levels of W^{S} are plotted in Fig. 3; they provide a geometrical representation of the state of the tensor perturbations in phase space and illustrate the meaning of the squeezing parameters: the ellipse representing the $\sqrt{2}-\sigma$ contour has semi-minor and semi-major axes of length $A_k = \sqrt{\hbar} e^{r_k}$ and $B_k = \sqrt{\hbar} e^{-r_k}$, which are tilted by the angle φ_k in phase space. The fluctuations of the operator in the direction of the semi-major axis are exponentially amplified with respect to the vacuum; this is called a super-fluctuant mode. On the other hand, the fluctuations of the operator related to the semi-major axis are exponentially suppressed, thus defining a sub-fluctuant mode.

图 3 绘制了 W^{S} 的部分等高线；它们给出了张量扰动态在相空间中的几何表示，也阐明了压缩参数的物理意义: 代表 $\sqrt{2}-\sigma$ 等高线的椭圆半短轴和半长轴长度分别为 $A_k = \sqrt{\hbar} e^{r_k}$ 和 $B_k = \sqrt{\hbar} e^{-r_k}$ ，在相空间中倾斜了角度 φ_k 。半长轴方向上算符的涨落相比真空发生了指数放大，这样的模式称为超涨落模式。而半短轴方向对应算符的涨落则发生了指数压制，因此定义为亚涨落模式。

Fig. 3$\sqrt{2}-\sigma$ contour level of the Wigner function W^{S} for $\varphi_k = \pi/4, r_k = 1$ (green ellipse) and the vacuum state $r_k = 0$ (pink circle). This figure is adapted from Ref. [46])

图 3$\sqrt{2}-\sigma$ 维格纳函数 W^{S} 对应 $\varphi_k = \pi/4, r_k = 1$ 的等高线 (绿色椭圆) 与真空态 $r_k = 0$ 的等高线 (粉色圆形)。本图改编自文献 [46])



The presence of amplification and suppression is a manifestation of the existence of a growing and a decaying solution in Eq. (4) [45]. Their complementarity can be traced back to the purity of the state which, for a Gaussian state, can be computed directly in terms of the covariance matrix via [43]

涨落的放大与压制正是式 (4) 中增长解和衰减解存在的体现 [45]。二者的互补性可以追溯到态的纯性，对于高斯态，纯性可以通过协方差矩阵按如下方式直接计算得到 [43]

$$p_k = \mathrm{tr}\left(\hat{\rho}^2\right) = \frac{1}{\sqrt{\det(\gamma)}} = \frac{1}{\gamma_{11}\gamma_{22} - \gamma_{12}^2} = \frac{\hbar^2}{A_k^2 B_k^2} = \frac{\hbar^2\pi^2}{S_k^2}, \tag{68}$$

where S_k is the area of the $\sqrt{2}-\sigma$ contour defined by the points at which the argument of the exponential in Eq. (57) is unity. Since the purity of the state is preserved under Hamiltonian evolution, so is S_k . Therefore, the amplification in a given direction has to be balanced out with squeezing in another. Conversely, if the fluctuations in one direction are reduced, they increase in another. For any quantum state, $p_k \leq 1$, and so the area is minimal for a pure state $p_k = 1$, like the one we consider here, where $S_k = \pi\hbar$; this is a geometrical translation of the Heisenberg uncertainty principle forbidding to localise the system too precisely. Note that in general, due to the rotation φ_k , the product uncertainty of the original pair $(\hat{\mu}_k, \hat{\pi}_k)$ does not saturate the inequality anymore.

其中 S_k 是由式 (57) 指数幅角为 1 的点所定义的 $\sqrt{2}-\sigma$ 轮廓的面积。由于哈密顿演化过程中态的纯度保持不变，因此 S_k 也保持不变。由此，某一方向上的放大必然会被另一方向的压缩抵消；反之，若某一方向的涨落减小，另一方向的涨落就会增大。对于任意量子态，都有 $p_k \leq 1$ ，因此纯态 $p_k = 1$ 的面积最小，就像本文研究的这种情况，此时 $S_k = \pi\hbar$ ；这是海森堡不确定性原理的几何表述，该原理不允许对系统进行过于精确的定位。注意一般情况下，由于旋转 φ_k ，原始对 $(\hat{\mu}_k, \hat{\pi}_k)$ 的乘积不确定性不再满足不等式饱和条件。

In addition to granting an elegant geometrical representation of the state, the presentation in terms of two-mode squeezed states is often used in the literature to discuss the quantumness of primordial gravitational waves and scalar perturbations alike. These aspects are discussed in section "Quantum Features in Primordial Gravitational Waves?."

除了能为量子态提供简洁的几何表示外，这种双模压缩态的表述在文献中常被用来讨论原初引力波和标量扰动的量子特性。这些内容将在“原初引力波中的量子特征?”一节中讨论。

Particle Production

粒子产生

Having laid out the formalisms to follow the evolution of gravitational waves in cosmology, we want to give more physical insights into the evolution and show that, under certain conditions, it can be understood as a process of particle creation. Bogoliubov transformations and mode functions are the appropriate way to describe this process in curved spacetime. We start by analysing their relation to particle content.

在搭建了追踪宇宙学中引力波演化的 formalism 后，我们希望为该演化提供更具物理意义的阐释，并说明在特定条件下，这一过程可以被理解为粒子产生过程。博戈留博夫变换和模函数是弯曲时空中描述该过程的合适工具。我们首先分析它们与粒子自由度的关系。

Consider two pairs of operators $(\hat{a}, \hat{a}^\dagger)$ and $(\hat{b}, \hat{b}^\dagger)$ related by a constant Bogoliubov transformation

考虑由常数博戈留博夫变换联系起来的两对算符 $(\hat{a}, \hat{a}^\dagger)$ 和 $(\hat{b}, \hat{b}^\dagger)$

$$\hat{b} = \alpha\hat{a} + \beta\hat{a}^\dagger, \tag{69}$$

with $(\alpha, \beta) \in \mathbb{C}^2$ such that $|\alpha|^2 - |\beta|^2 = 1$. We define two vacua: $|0\rangle_a$ with respect to the $\hat{a}$ operators and $|0\rangle_b$ with respect to the $\hat{b}$ operators. The crucial observation is that these vacua do not coincide. The number of b -particles in the a -vacuum is always non-vanishing when the Bogoliubov transformation is non-trivial

其中满足 $(\alpha, \beta) \in \mathbb{C}^2$ 使得 $|\alpha|^2 - |\beta|^2 = 1$ 。我们定义两个真空: 对应 $\hat{a}$ 算符的 $|0\rangle_a$ ， 以及对应 $\hat{b}$ 算符的 $|0\rangle_b$ 。核心结论是这两个真空并不等价。当博戈留博夫变换非平凡时，a 真空中的 b 粒子数总是非零的

$${}_a\langle 0 \left| \hat{b}^\dagger \hat{b} \right| 0\rangle_a = |\beta|^2 > 0. \tag{70}$$

The analysis carries over to the study of $\pm\mathbf{k}$ modes. Equation (46) shows that the vacuum of the operators $\hat{a}_{\pm\mathbf{k}}(\eta)$ is filled with particles associated to $\hat{a}_{\pm\mathbf{k}}(\eta_{\rm in})$. We thus already see that the number of particles will be different for the same state using the operators $\hat{a}_{\pm\mathbf{k}}(\eta)$ of Eq. (28) at two different times.

上述分析可以直接推广到对 $\pm\mathbf{k}$ 模的研究。式 (46) 表明，算符 $\hat{a}_{\pm\mathbf{k}}(\eta)$ 的真空会充满与 $\hat{a}_{\pm\mathbf{k}}(\eta_{\rm in})$ 关联的粒子。因此我们不难看出，对于同一个量子态，使用式 (28) 的算符 $\hat{a}_{\pm\mathbf{k}}(\eta)$ 在两个不同时刻计算得到的粒子数是不同的。

What are then the appropriate operators to describe the particle content of the field $\hat{\mu}$ and define a vacuum as we have in section "Squeezed States?" We have so far considered operators defined by Eq. (28). In Minkowski spacetime $(a' = 0)$, this form is uniquely selected (up to a phase) by requiring that the Hamiltonian is diagonal and that the vacuum thus defined is invariant under the Poincaré group so that it is shared by all inertial observers, or, equivalently, the vacuum is the ground state of the Hamiltonian [34]. In this situation there is a preferred set of operators selected by physical symmetries, which subsequently define preferred notions of vacuum and particle.

那么，描述场 $\hat{\mu}$ 的粒子自由度并定义真空的合适算符是什么呢？就像我们在"压缩态"小节中做的那样，目前我们已经研究了由式 (28) 定义的算符。在闵氏时空 $(a' = 0)$ 中，要求哈密顿量对角化，且由此定义的真空满足庞加莱群不变性 (从而被所有惯性观测者共享)，或者等价地要求该真空是哈密顿量的基态 [34]，就能唯一确定 (相差一个相位不影响) 这种算符形式。这种情况下，物理对称性选出了一组优先的算符，进而定义了优先的真空和粒子概念。

The procedure described above breaks down in an expanding Universe, $a' \neq 0$, as the Poincaré group is no longer a symmetry of spacetime, ω_k^2 [see Eq. (27)] is time-dependent and can even become negative so that the existence of an energy minimum is not guaranteed anymore. We are left with no physically preferred vacuum in which no inertial detector would record the presence of particles. In this context, the choice of $\hat{a}_{\mathbf{k}}(\eta_{\rm in})$ to perform the expansion (38) appears arbitrary.

上文所述的流程在膨胀宇宙中不再成立，$a' \neq 0$：因为庞加莱群不再是时空的对称性，ω_k^2 [见式 (27)] 是含时的，甚至可能变为负值，因此无法保证存在能量最小值。我们不再拥有物理上优先的真空——不存在能让惯性探测器不记录到任何粒子的真空。在这个背景下，用于展开式 (38) 的 $\hat{a}_{\mathbf{k}}(\eta_{\rm in})$ 选择看起来是任意的。

A choice of operators in fact corresponds to a choice of mode functions, the latter being more convenient to work with. Consider the operators $\hat{b}_{\pm\mathbf{k}}$ related to $\hat{a}_{\pm\mathbf{k}}(\eta_{\rm in})$ by the following time-independent Bogoliubov transformation

事实上，算符的选择对应模函数的选择，而模函数处理起来更方便。考虑通过如下不含时博戈留博夫变换与 $\hat{a}_{\pm\mathbf{k}}(\eta_{\rm in})$ 关联的算符 $\hat{b}_{\pm\mathbf{k}}$

$$\hat{b}_{\mathbf{k}} = \rho_k^\star \hat{a}_{\mathbf{k}}(\eta_{\rm in}) + \chi_k \hat{a}^\dagger_{-\mathbf{k}}(\eta_{\rm in}), \tag{71}$$

with $(\rho_k, \chi_k) \in \mathbb{C}^2$ such that $|\rho_k|^2 - |\chi_k|^2 = 1$. Inverting this transformation and inserting in Eq. (38), we get

其中满足 $(\rho_k, \chi_k) \in \mathbb{C}^2$ 使得 $|\rho_k|^2 - |\chi_k|^2 = 1$ 。对该变换求逆后代入式 (38)，我们得到

$$\hat{\mu}(\mathbf{x}, \eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}.\mathbf{x}} v_k(\eta) \hat{b}_{\mathbf{k}} + e^{-i\mathbf{k}.\mathbf{x}} v_k^\star(\eta) \hat{b}_{\mathbf{k}}^\dagger \right],$$

where

其中

$$v_k = \rho_k^\star u_k - \chi_k u_k^\star \tag{72}$$

can be checked to be a mode function, i.e., a solution of Eq. (4) with a Wronskian normalised to $W\left(v_k, v_k^\star\right) = \left(|\rho_k|^2 - |\chi_k|^2\right) W\left(u_k, u_k^\star\right) = -i\hbar$. A similar expansion is found for $\hat{\pi}$, with v_k replaced by new functions V_k defined as the U_k s through the replacement $u_k \to v_k$.

可以验证它是一个模函数，即满足朗斯基行列式归一化为 $W\left(v_k, v_k^\star\right) = \left(|\rho_k|^2 - |\chi_k|^2\right) W\left(u_k, u_k^\star\right) = -i\hbar$ 的式 (4) 的解。对 $\hat{\pi}$ 可得到类似展开，只需将 v_k 替换为通过替换 $u_k \to v_k$ 定义的新函数 V_k (即 U_k)。

We then have an alternative expansion of $\hat{\mu}$ and $\hat{\pi}$ over another set of mode functions and operators. The meaning of the operators in the expansion is set once the associated mode functions are fixed. This can be seen by expressing $\hat{b}_{\mathbf{k}}$ in terms of the mode function and the fields $\hat{b}_k = -i\left[V_k^\star \hat{\mu}(\mathbf{k}, \eta) - v_k^\star \hat{\pi}(\mathbf{k}, \eta)\right]$. A choice of mode functions corresponds to a choice of initial conditions for the solutions of Eq. (4). The normalisation of the Wronskian fixes one condition, and one is left to choose. For instance, the Minkowski operators (28) are associated to the mode function

我们随后得到了 $\hat{\mu}$ 和 $\hat{\pi}$ 在另一组模函数和算符下的等价展开。展开中算符的含义在关联模函数确定后就固定了，这可以通过将 $\hat{b}_{\mathbf{k}}$ 用模函数和场 $\hat{b}_k = -i\left[V_k^\star \hat{\mu}(\mathbf{k}, \eta) - v_k^\star \hat{\pi}(\mathbf{k}, \eta)\right]$ 表示看出。选择模函数对应为式 (4) 的解选择初条件，朗斯基行列式的归一化确定了一个条件，还剩一个条件需要选择。例如，闵氏算符 (28) 就关联如下模函数

$$u_k^{(\mathrm{M})}(\eta) = \sqrt{\frac{\hbar}{2k}} e^{-ik\eta}, \tag{73}$$

corresponding to the initial conditions

对应初条件

$$u_k(\eta_0) = \sqrt{\frac{\hbar}{2k}} e^{-ik\eta_0} \quad \text{and} \quad u_k'(\eta_0) = -i\sqrt{\frac{\hbar k}{2}} e^{-ik\eta_0}. \tag{74}$$

For non-vanishing $\mathcal{H}, u_k^{(\mathrm{M})}$ is no longer a solution of Eq. (4). Yet, when analysing the evolution of the two helicities of the gravitational field in this context, we have used the associated operators (28). Their time dependence then does not simply factorise in the running phase of $u_k^{(\mathrm{M})}$, and we have to deal with a continuous change of reference operators parametrised by a time-dependent

当 $\mathcal{H}, u_k^{(\mathrm{M})}$ 非零时，它不再是式 (4) 的解。但在此框架下分析引力场两种螺旋度的演化时，我们仍使用关联算符 (28)。它们的时间依赖无法简单分解到 $u_k^{(\mathrm{M})}$ 的流动相位中，因此我们必须处理由含时参数描述的参考算符的连续变化

Bogoliubov transformation. These operators correspond at any time η to what would be the Minkowskian definition of particle and vacuum if the modulation were to stop at this instant. Alternatively, we can work with the operators defined at some fixed time η_{in}, as we did in Eq. (36), in which case the time dependence is that of a mode function satisfying Eq. (4) which differs from that of $u_k^{(\mathrm{M})}$. As just discussed, when the background is time-dependent, neither of these two sets of operators can be favoured to discuss the particle content of the field.

即 Bogoliubov 变换。对任意时刻 η，若调制在该时刻停止，这些算符就对应粒子和真空的闵氏定义。或者，我们也可以像在式 (36) 中那样，使用某个固定时刻 η_{in} 定义的算符，此时时间依赖来自满足式 (4) 的模函数，与 $u_k^{(\mathrm{M})}$ 的时间依赖不同。正如前文讨论，当背景含时时，这两组算符都没有优势来讨论场的粒子内容。

There are some situations where one can unambiguously define particles and their properties. One such case [17] is that of a spacetime which is asymptotically Minkowski at both very early and very late times, i.e., one for which the scale factor varies in between two asymptotic constant values

存在一些可以无歧义定义粒子及其性质的情况。其中一种情况 [17] 是时空在极早期和极晚期都渐近闵氏，即标度因子在两个渐近常数之间变化

$$a(\eta) \underset{\eta\to-\infty}{\to} a_{\text{in}} \text{ and } a(\eta) \underset{\eta\to+\infty}{\to} a_{\text{out}}.$$

We can therefore define asymptotically Minkowski "in" and "out" mode functions $u_k^{(\text{in/out})}$ and associated operators $\hat{a}_k^{(\text{in/out})}$ by requiring as initial condition that they match the Minkowski solution

因此我们可以定义渐近闵氏的“in”和“out”模函数 $u_k^{(\text{in/out})}$ 与关联算符 $\hat{a}_k^{(\text{in/out})}$，要求它们作为初条件匹配闵氏解

$$u_k^{(\text{in})} \underset{\eta\to-\infty}{\to} \sqrt{\frac{\hbar}{2k}} e^{-ik\eta} \text{ and } u_k^{(\text{out})} \underset{\eta\to+\infty}{\to} \sqrt{\frac{\hbar}{2k}} e^{-ik\eta}.$$

These mode functions are both solution of Eq. (4) for any time η and are therefore related by a time-independent Bogoliubov transformation

这些模函数对任意时刻 η 都是式 (4) 的解，因此它们之间由不随时间变化的 Bogoliubov 变换联系

$$u_k^{(\text{in})} = \rho_k u_k^{(\text{out})} + \chi_k u_k^{\star(\text{out})}, \tag{75}$$

and, via Eq. (71), so are $\hat{a}_k^{(\text{in/out})}$, and it is straightforward to evaluate the number of particles produced by the non-trivial evolution of the background. We assume that the field is initially (in the "in" region) in the vacuum defined by the "in" operators where there exists a preferred notion of vacuum; we denote $|0\rangle_{\text{in}}$ this "in" vacuum. In order to read the particle content at the end of evolution (in the "out" region), we need to use the ”out” operators that define the Minkowski notion of particle there. The number of particles in the ”out” region is given by

并且借助式 (71)，$\hat{a}_k^{(\text{in/out})}$ 也同样满足，因此可以很直接地计算出背景非平凡演化产生的粒子数。我们假设场初始 (“入”区) 处于由“入”算符定义的真空，该真空存在一个优先的真空概念；我们将此“入”真空记为 $|0\rangle_{\text{in}}$ 。为了得到演化结束 (“出”区) 的粒子数，我们需要使用在该处定义闵氏粒子概念的“出”算符。“出”区的粒子数由下式给出

$$n_{\pm\mathbf{k}}^{\text{out}} = {}_{\text{in}}\langle 0 \left| \hat{a}_{\pm\mathbf{k}}^{\dagger(\text{out})} \hat{a}_{\pm\mathbf{k}}^{(\text{out})} \right| 0 \rangle_{\text{in}} = |\chi_k|^2. \tag{76}$$

This number is strictly positive and the same in the modes $\pm\mathbf{k}$; this is the wellknown phenomenon of pair production out of the vacuum, here powered by the background expansion. To evaluate the extent of this production quantitatively, we have to compute the mode equation for both ”in” and ”out” conditions and match them. This computation can, for example, be done exactly in a 2D model where the scale factor evolves as a hyperbolic tangent between its asymptotic values [47].

该粒子数严格为正，且对所有模式 $\pm\mathbf{k}$ 都相同；这就是真空对产生这一众所周知的现象，在此处由背景膨胀驱动。要定量计算该产生过程的强度，我们必须分别对“入”和“出”条件求解模式方程并对它们进行匹配。例如，当标度因子在渐近值之间按双曲正切规律演化时，该计算可以在二维模型中精确完成 [47]。

Let us make the connection in this idealised case with the time-dependent Bogoliubov coefficients solving the dynamics (33). First, note that the operators

在此理想化情形中，我们将其与求解动力学的含时博戈留波夫系数建立联系 (33)。首先，注意到这些算符

(28) coincide with those defined with respect to a mode function u_k at times η_0 where it satisfies the Minkowski conditions (74). This can be checked directly upon inserting Eq. (28) in the expression of the operator in terms of the mode function and the fields at time η_0 . This applies in both the ”in” and ”out” regions

(28) 与相对于模式函数 u_k 在时刻 η_0 定义的算符一致，在 η_0 处模式函数满足闵氏条件 (74)。将式 (28) 代入由时刻 η_0 的模式函数和场表示的算符表达式中，即可直接验证这一点，该结论对“入”区和“出”区均成立

$$\hat{a}_{\pm\mathbf{k}}(\eta) \xrightarrow[\eta\to-\infty]{} \hat{a}_{\pm\mathbf{k}}^{(\text{in})},$$

$$\hat{a}_{\pm\mathbf{k}}(\eta) \xrightarrow[\eta\to+\infty]{} \hat{a}_{\pm\mathbf{k}}^{(\text{out})}.$$

The time-independent Bogoliubov coefficients between the ”in” and the ”out” states, therefore, correspond to the late time limit of the time-dependent Bogoliubov coefficients of Eq. (34)

因此，“入”态与“出”态之间的不含时博戈留波夫系数，对应式 (34) 中含时博戈留波夫系数的迟时间极限

$$\rho_k = \alpha_k(\eta \to +\infty) \quad \text{and} \quad \chi_k = \beta_k(\eta \to +\infty), \tag{77}$$

where the associated number of particles $\left\langle \hat{a}^\dagger_{\pm\mathbf{k}}(\eta)\,\hat{a}_{\pm\mathbf{k}}(\eta)\right\rangle$ and correlations are now meaningful. While it is not a priori the case at any intermediate times, since the scale factor is varying, we discuss in section ”Connection to Observations” how it is often possible to identify ”in” and ”out” regions for certain ranges of modes k in the cosmological evolution.

其中关联的粒子数 $\left\langle \hat{a}^\dagger_{\pm\mathbf{k}}(\eta)\,\hat{a}_{\pm\mathbf{k}}(\eta)\right\rangle$ 和关联现在是有意义的。虽然由于标度因子在变化，任意中间时刻一般不满足该性质，但我们会在“与观测的联系”一节讨论，在宇宙学演化中，对于一定范围的模式 k ，通常总能区分出“入”区和“出”区。

Anticipating these considerations, we conclude by making a connection with section ”Squeezed States” and studying the particle content of a two-mode squeezed state. Those can also be fully characterised by the following three non-vanishing expectation values (two of them being equal)

结合这些考虑，我们最后将其与“压缩态”一节建立联系，研究双模压缩态的粒子内容。双模压缩态也可以完全由以下三个非零期望表征 (其中两个相等)

$$n_k = \left\langle \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle = \left\langle \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\right\rangle = \frac{\gamma_{11}+\gamma_{22}-2}{4} = \sinh^2(r_k), \tag{78a}$$

$$c_k = \langle \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\rangle = \frac{\gamma_{11}-\gamma_{22}}{4} + i\frac{\gamma_{12}}{2} = -\frac{1}{2}\sinh(2r_k)\,e^{2i\varphi_k}. \tag{78b}$$

These expressions are obtained by inverting Eq. (28) and making use of Eqs. (65) and (66). The first expectation value n_k gives the number of particles in the modes $\mathbf{k}$ and $-\mathbf{k}$, which must be identical because of isotropy, while c_k encodes the two-mode coherence of the pairs. Imposing the purity to be less than unity, $p_k = \gamma_{11}\gamma_{22} - \gamma_{12}^2 \leq 1$, yields the following bound on the magnitude of this coherence

这些表达式通过反转式 (28) 并利用式 (65) 和 (66) 得到。第一个期望 n_k 给出模式 $\mathbf{k}$ 和 $-\mathbf{k}$ 中的粒子数，由各向同性可知二者必须相等，而 c_k 编码了对的双模相干性。要求纯度小于 1，即 $p_k = \gamma_{11}\gamma_{22} - \gamma_{12}^2 \leq 1$ ，可得到该相干性幅度的以下界

$$|c_k| \leq \sqrt{n_k(n_k+1)}\,. \tag{79}$$

For a pure state like that of the gravitons, the bound is saturated $|c_k| = \sqrt{n_k(n_k+1)}$, while for a thermal state $c_k = 0$. In this sense, the modes are uncorrelated in the thermal state and maximally correlated in a two-mode squeezed state; they are even entangled [44]. We come back to this important point in section ”Quantum Features in Primordial Gravitational Waves?”.

对于引力子这样的纯态，该界是紧的 $|c_k| = \sqrt{n_k(n_k+1)}$ ，而对于热态则是 $c_k = 0$ 。在此意义上，热态中的模式互不相关，而双模压缩态中的模式最大相关；它们甚至是纠缠的 [44]。我们会在“原初引力波中的量子特征?”一节回到这一重要问题。

Anomaly-Induced Semiclassical Theory

反常诱导半经典理论

The concept of particle associated with a quantum field is a global one in the sense that it is defined through modes; somehow, it can be understood, as described above, as the effect of geometry on matter, even when ”matter” consists of tensor-like perturbations of the gravitational field itself. When coupled to classical GR in a semiclassical way, the quantum nature of gravitational waves, just like any other particle, may also manifest itself in another way, namely, in the back reaction of their quantum fields on geometry (see, e.g., the historical papers by M. J. Duff [48-50] who proposed it for the first time and Refs. [51, 52] as well as the more recent Ref. [53]). This approach is, therefore, the opposite of the above, making extensive use of the stress-energy tensor $T_{\mu\nu}(x)$, which is a local quantity.

与量子场关联的粒子概念是全局性的，它通过模定义；某种程度上，如前文所述，这可以理解为几何对物质的效应，即便这里的“物质”本身就是引力场的类张量微扰。当以半经典方式耦合到经典广义相对论时，引力波的量子性质和其他任何粒子一样，也可以另一种方式显现，即其量子场对几何的反作用 (例如，参见 M. J. Duff 首次提出该想法的早期文献 [48-50]、文献 [51,52] 以及更近的文献 [53])。因此，该方法与前述方法相反，它大量使用应力-能量张量 $T_{\mu\nu}(x)$ ，这是一个局域量。

In this section, for the sake of notational simplicity, we set $\hbar \to 1$ as all the effects are quantum by nature.

在本节中，为简化记号，我们设定 $\hbar \to 1$ ，因为所有效应本质上都是量子的。

Gravity with Quantum Fields

含量子场的引力

When quantum fields are described in a geometric background, it is customary to write the corresponding Einstein’s equations in the semiclassical form

当在几何背景下描述量子场时，通常将对应的爱因斯坦方程写为半经典形式

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}}\langle T_{\mu\nu}\rangle_{\mathrm{ren}}, \tag{80}$$

so that geometry is now sourced by the renormalised stress-energy tensor $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$.

此时几何由重整化后的能量动量张量 $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$ 源生。

As the classical Einstein equations are derived from a variation of the vacuum Einstein-Hilbert term, possibly including a cosmological constant contribution (note that we do not consider the Gibbons-Hawking-York boundary term in these discussions; it can be set to zero by assuming a compact manifold.),

由于经典爱因斯坦方程由真空爱因斯坦-希尔伯特项变分得到，变分过程可能包含宇宙常数贡献 (注意我们在此讨论中不考虑吉本斯-霍金-约克边界项；可通过假设流形紧致将其设为零),

$$S_{\mathrm{EH}\Lambda} = \frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^4 x \sqrt{-g}\,(R + 2\Lambda)\,, \tag{81}$$

the stress-energy tensor being derived from the classical matter action $\mathcal{S}_{\mathrm{m}}$ through

能量动量张量由经典物质作用量 $\mathcal{S}_{\mathrm{m}}$ 通过以下方式导出

$$T_{\mu\nu}^{\mathrm{class}} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_{\mathrm{m}}}{\delta g^{\mu\nu}}, \tag{82}$$

one can recover the semiclassical case (80) by similarly defining an effective action $\Gamma\left[g_{\mu\nu}\right]$ such that

我们可以通过类似方法定义有效作用量 $\Gamma\left[g_{\mu\nu}\right]$，从而得到半经典情形 (80)，满足

$$\left\langle T_{\mu\nu} \right\rangle = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}}. \tag{83}$$

It can be shown that for a set of matter fields denoted generically by ϕ , and which can include scalar, gauge and fermion fields, whose dynamics is driven by the action $S\left[\phi; g_{\mu\nu}\right]$, one finds

可以证明，对于一组以 ϕ 通用标记的物质场，其中可包含标量场、规范场和费米子场，其动力学由作用量 $S\left[\phi; g_{\mu\nu}\right]$ 描述，可得

$$e^{i\Gamma[g_{\mu\nu}]} = \int \mathcal{D}\phi\, e^{iS[\phi; g_{\mu\nu}]}, \tag{84}$$

and the expectation value in Eq. (83) is then understandable in terms of ”in” and ”out” vacuum states

式 (83) 中的期望可以通过“入”真空态和“出”真空态来理解

$$\frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}} = \frac{{}_{\mathrm{out}}\left\langle 0 \left| T_{\mu\nu} \right| 0 \right\rangle_{\mathrm{in}}}{{}_{\mathrm{out}}\left\langle 0 \mid 0 \right\rangle_{\mathrm{in}}}, \tag{85}$$

thereby automatically providing the required normalisation.

由此自动给出所需的归一化。

In order to integrate explicitly Eq. (85) and obtain the relevant effective action, one needs to know the matter content and its corresponding action. Compared to their flat space counterparts, fermionic and vectorial contributions are merely obtained by the minimal coupling, namely, making the replacements $\partial \to \nabla$ and using the metric $g_{\mu\nu}$ to integrate. The scalar field case can also include an extra term, not present in the flat Minkowski situation, and one gets

为了显式积分式 (85) 得到相关有效作用量，需要知道物质内容及其对应作用量。与平直空间的结果相比，费米子和矢量的贡献仅需通过最小耦合得到，即进行替换 $\partial \to \nabla$ 并使用度规 $g_{\mu\nu}$ 进行积分。标量场还可以包含一项平直闵氏空间中不存在的额外项，得到

$$S_\varphi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[(\partial\varphi)^2 + \xi_{ij}\varphi^i\varphi^j R\right], \tag{86}$$

where we considered a set of scalars $\{\varphi^i\} = \boldsymbol{\varphi}$; a possible extra potential term $V(\boldsymbol{\varphi})$ can be added to this action. Equation (86) involves a set of new dimensionless numbers $\{\xi_{ij}\}$ which are called non-minimal parameters. For a single scalar field, this reduces to a single parameter; its special value $\xi = \frac{1}{6}$ yields conformal invariance.

其中我们考虑了一组标量场 $\{\varphi^i\} = \boldsymbol{\varphi}$；还可在该作用量中添加一个额外的势能项 $V(\boldsymbol{\varphi})$ 。式 (86) 包含一组新的无量纲数 $\{\xi_{ij}\}$，称为非最小参数。对于单个标量场，这退化为单个参数；当其取特殊值 $\xi = \frac{1}{6}$ 时，理论具有共形不变性。

It turns out that the action derived from this procedure contains ultraviolet divergences that thus need to be renormalised. These lead to contributions that are purely geometrical, involving only scalars made out of the Riemann tensor $R_{\mu\nu\alpha\beta}(x)$ and its contractions. This is understandable as short wavelengths are only sensitive to local features of spacetime. Regularising and renormalising forces to introduce counterterms involving higher-order derivatives, one is naturally led to the conclusion that in order to obtain a renormalisable theory of quantum matter on a classical curved spacetime, one must demand a geometrical framework that goes beyond general relativity.

事实证明，通过该过程得到的作用量存在紫外发散，因此需要进行重整化。这些发散会带来仅由黎曼张量 $R_{\mu\nu\alpha\beta}(x)$ 及其缩并构成的标量所对应的纯几何贡献。这是合理的，因为短波长仅对时空的局域特征敏感。正则化与重整化要求引入包含高阶导数的抵消项，因此自然可以得到结论: 为了得到经典弯曲背景下量子物质的可重整化理论，我们需要一个超出广义相对论范围的几何框架。

Applying the procedure described above, the relevant vacuum classical action

应用上述流程，相关的真空经典作用量

$$S_{\text{vac}} = S_{\text{EH}\Lambda} + S_{\text{HD}} \tag{87}$$

is found to include the usual Einstein-Hilbert term (81) in which both G_{N} and Λ are renormalised quantities, but another contribution, containing higher derivative (HD) terms, needs to be included, namely,

被发现包含通常的爱因斯坦-希尔伯特项 (81)，其中 G_{N} 和 Λ 都是重整化后的量，此外还需要额外添加一项包含高阶导数 (HD) 的贡献，即:

$$S_{\text{HD}} = \int \mathrm{d}^4x\sqrt{-g}\left(a_1C^2 + a_2E + a_3\Box R + a_4R^2\right), \tag{88}$$

where

其中

$$C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + \frac{1}{3}R^2$$

is the square of the Weyl tensor and

是外尔张量的平方，且

$$E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$$

represents the Gauss-Bonnet topological term. The action (87) has been shown [54] to lead to a renormalisable (albeit containing unphysical ghosts or having non-unitarity issues) theory of quantum gravity. Details can be found in particular Ref. [55] in the present volume. The parameter a_3 is irrelevant for the equations of motion since $\Box R$ is a surface term, while the R^2 term is at the origin of the most serious inflation model proposed by Starobinsky [56].

代表高斯-博内拓扑项。文献 [54] 已证明，作用量 (87) 可导出一个可重整化的量子引力理论——尽管该理论包含非物理鬼场，且存在非幺正性问题。具体细节可参见本卷中的文献 [55]。参数 a_3 对方程运动没有影响，因为 $\Box R$ 是一个边界项，而 R^2 项是斯塔罗宾斯基提出的最经典的暴胀模型的起源 [56]。

Conformal Anomalies

共形反常

Let us consider a conformally invariant theory, i.e., for which the transformations

我们来研究共形不变理论，即在该理论中，变换

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu} \ \text{ and } \ \varphi \to \varphi/\Omega(x) \tag{89}$$

(vector fields being left unchanged and spinors transforming with $\Omega^{-3/2}$) leave the action S unchanged. From this requirement, one finds that the trace of the energy-momentum tensor [51]

(矢量场保持不变，旋量按 $\Omega^{-3/2}$ 变换) 让作用量 S 保持不变。根据这一要求，可以得到能量动量张量的迹 [51]

$$T^\mu_\mu\left[g_{\alpha\beta}(x)\right] = -\frac{\Omega(x)}{\sqrt{-g(x)}}\frac{\delta S\left[\bar{g}_{\mu\nu}\right]}{\delta\Omega(x)}\bigg|_{\Omega\to 1}, \tag{90}$$

should vanish if the transformation (89) represent a symmetry of S . This implies that the scalar fields are massless and $\xi \to \frac{1}{6}$. The identity (90) is true at the classical level, and indeed the conserved Noether current in this case reads

若变换 (89) 是 S 的对称性，则迹应当为零。这说明标量场是无质量的，且满足 $\xi \to \frac{1}{6}$ 。等式 (90) 在经典层面成立，此时守恒的诺特流确实可以写为

$$\left(2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \sum_i k_i\phi_i\frac{\delta}{\delta\phi_i}\right)S\left[g_{\alpha\beta}(x),\phi(x)\right] = 0, \tag{91}$$

in which the weights k_i correspond to the various fields involved, with $k_\mathrm{s} = -1$ for scalar fields, $k_\mathrm{f} = -3/2$ for the fermions and $k_\mathrm{v} = 0$ for the gauge fields.

其中权重 k_i 对应涉及的各类场，标量场对应 $k_\mathrm{s} = -1$ ，费米子对应 $k_\mathrm{f} = -3/2$ ，规范场对应 $k_\mathrm{v} = 0$ 。

At the quantum level, however, the trace $\langle T^\mu\rangle$ is no longer vanishing, as explicitly calculating it with the given matter content (scalar, vector and spinor fields) yields a renormalised expectation value [51]

但在量子层面，迹 $\langle T^\mu\rangle$ 不再为零，对给定物质组分 (标量、矢量和旋量场) 进行显式计算后，可以得到重整化期望值 [51]

$$\left\langle T^\mu_\mu\right\rangle = -\left(\omega C^2 + bE + c\Box R\right), \tag{92}$$

where the β -functions ω, b and c depend on the numbers of real scalar degrees of freedom N_0 , four-component spinor fermions $N_{1/2}$ and vector fields N_1 in the underlying particle physics model. In practice, they are found to be

其中 β 函数 ω, b 和 c 依赖于基础粒子物理模型中实标量自由度数目 N_0 、四分量旋量费米子数目 $N_{1/2}$ 和矢量场数目 N_1 。实际计算得到它们为

$$\begin{pmatrix}\omega\\ b\\ c\end{pmatrix} = \frac{1}{360(4\pi)^2}\begin{pmatrix}3N_0 + 18N_{1/2} + 36N_1\\ -N_0 - 11N_{1/2} - 62N_1\\ 2N_0 + 12N_{1/2} - 36N_1\end{pmatrix}. \tag{93}$$

In the standard model (SM) of particle physics, where the SU(3)×SU(2)×U(1) is broken to SU(3)×U(1) through a Higgs doublet, the relevant numbers are $N_0^\mathrm{SM} = 4$, $N_1^\mathrm{SM} = 12$ (eight gluons, the intermediate $W^\pm$ and Z^0 and the photon) and $N_{1/2}^\mathrm{SM} = 24$ (leptons and quarks, assuming a massive neutrino), one finds

在粒子物理的标准模型 (SM) 中，SU(3) × SU(2) × U(1) 通过希格斯二重态破缺为 SU(3) × U(1) ，相关的数目为 $N_0^\mathrm{SM} = 4$ 、$N_1^\mathrm{SM} = 12$ (8 个胶子、中间 $W^\pm$ 、Z^0 和光子) 以及 $N_{1/2}^\mathrm{SM} = 24$(轻子和夸克，假设中微子有质量)，可以得到

$$\omega^\mathrm{SM} = \frac{73}{480\pi^2},\ b^\mathrm{SM} = -\frac{253}{1440\pi^2}\ \text{ and }\ c^\mathrm{SM} = -\frac{17}{720\pi^2}.$$

Note that although b is negative definite, the sign of c depends on the exact matter content: measuring this sign somehow, e.g., through that of the primordial gravitational wave spectrum, could be an indirect way of getting information about the physics that should apply at high energies such as the grand unification (if any) scale. Note, for instance, that in the case of the minimal supersymmetric extension of the standard model (MSSM), the number of vector modes is unchanged $\left(N_1^\mathrm{MSSM} = 12\right)$, while the number of fermions is increased to $N_{1/2}^{\rm MSSM}=32$ and the proliferation of new scalar modes then yields $N_0^{\rm MSSM}=104$, leading to $c^{\rm MSSM}=1/\left(36\pi^2\right)>0$.

请注意，尽管 b 是负定的，c 的符号依赖于具体的物质组分: 通过某种方式测量该符号 (例如通过原初引力波谱的符号)，可以间接获取高能标物理 (若存在大统一，则包括大统一标度) 的信息。例如，在最小超对称标准模型 (MSSM) 中，矢量模式的数目不变 $\left(N_1^{\rm MSSM}=12\right)$，但费米子数目增加到 $N_{1/2}^{\rm MSSM}=32$，新增标量模式后得到 $N_0^{\rm MSSM}=104$，最终结果为 $c^{\rm MSSM}=1/\left(36\pi^2\right)>0$.

Integrating the trace of (83) using Eq. (92) is a non-trivial task that has been achieved in Refs. [57, 58]. Ref. [59] suggested to rewrite the action in terms of two auxiliary scalar fields σ and ρ (see also Ref. [60] for an independent but equivalent formulation) which happens to be particularly useful for the gravitation wave discussion. It reads

利用式 (92) 对 (83) 的迹积分是一项非平凡工作，文献 [57,58] 已经完成了这项工作。文献 [59] 建议用两个辅助标量场 σ 和 ρ 重写作用量 (独立且等价的表述也可参见文献 [60])，该形式对引力波的讨论尤其方便。形式为

$$\begin{aligned}\Gamma = {} & S_{\rm c}\left[g_{\mu\nu}\right] + \int {\rm d}^4x\sqrt{-g}\left(\frac{1}{2}\sigma\Delta_4\sigma - \frac{1}{2}\rho\Delta_4\rho + \ell_1 C^2\rho\right) \\ & + \int {\rm d}^4x\sqrt{-g}\left\{\sigma\left[k_1C^2 + k_2\left(E - \frac{2}{3}\Box R\right)\right] - \frac{1}{12}k_3R^2\right\},\end{aligned} \tag{94}$$

where the integration constant $S_{\rm c}\left[g_{\mu\nu}\right]$ is conformally invariant, the covariant conformal fourth-order operator is (see Refs. [57,58])

其中积分常数 $S_{\rm c}\left[g_{\mu\nu}\right]$ 是共形不变的，协变共形四阶算符为 (参见文献 [57,58])

$$\Delta_4 = \Box^2 + 2R^{\mu\upsilon}\nabla_\mu\nabla_\upsilon - \frac{2}{3}R\Box + \frac{1}{3}R^{;\mu}\nabla_\mu,$$

and the coefficients are given in terms of those of Eq. (92) through

且系数可以通过式 (92) 的系数表示为

$$k_1 = -\frac{\omega}{2\sqrt{|b|}},\ k_2 = \frac{\sqrt{|b|}}{2},\ k_3 = c + \frac{2}{3}b \text{ and } \ell_1 = \frac{\omega}{2\sqrt{|b|}} \tag{95}$$

(recall $b<0$). This effective action stemming from the conformal anomaly (the Noether current is not conserved at the quantum level) should be added to the vacuum term $S_{\rm vac}$ of Eq. (87).

(回顾 $b<0$)。这个由共形反常导出的有效作用量 (诺特流在量子层面不守恒) 应当添加到式 (87) 的真空项 $S_{\rm vac}$ 中。

Anomaly-Induced Cosmology and Gravitational Waves

反常诱导宇宙学与引力波

Let us apply the above discussion to the specific case of a cosmological framework which is our main subject, first by considering a background FLRW (conformally flat) solution and its tensorial perturbations.

下面我们将上述讨论应用到作为我们研究主题的宇宙学框架这一具体情形,首先考虑背景 FLRW(共形平坦) 解及其张量扰动。

The FLRW metric can be written as a conformal transformation of the Minkowski metric $\eta_{\mu\nu}$ by setting $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$. In this very simple case, variations of the action (94) with respect to the auxiliary fields σ and ρ yield

FLRW 度规可以写为闵氏度规 $\eta_{\mu\nu}$ 的共形变换，通过设定 $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ 得到。在这个极简单的情形中，作用量 (94) 对辅助场 σ 和 ρ 变分得到

$$\left(\partial_t^2 - \nabla^2\right)\left(\sigma + 8\pi\sqrt{|b|}\ln a\right) = 0 \text{ and } \left(\partial_t^2 - \nabla^2\right)\rho = 0, \tag{96}$$

with solutions

其解为

$$\sigma = \sigma_{\rm h} - 8\pi\sqrt{|b|}\ln a \quad \text{and} \quad \rho = \rho_{\rm h}, \tag{97}$$

in which $\sigma_{\rm h}$ and $\rho_{\rm h}$ are solutions of the homogeneous equation, $\left(\partial_t^2 - \nabla^2\right) f_{\rm h} = 0$; they can be set to zero in the cosmological context. In this case, one finds the relation

其中 $\sigma_{\rm h}$ 和 $\rho_{\rm h}$ 是齐次方程的解，$\left(\partial_t^2 - \nabla^2\right) f_{\rm h} = 0$；在宇宙学背景下可将它们设为零。由此可得关系

$$\frac{{\rm d}^n\sigma}{{\rm d}t^n} = -8\pi\sqrt{|b|}\frac{{\rm d}^{n-1}H}{{\rm d}t^{n-1}},$$

where $H = \dot{a}/a$.

其中 $H = \dot{a}/a$ 。

The above solution (97) with the FLRW metric can now be inserted into the full theory containing both the action (94) and the original vacuum (87). It leads to the modified Friedmann equation

现在可将上述带有 FLRW 度规的解 (97) 代入同时包含作用量 (94) 和原真空作用量 (87) 的完整理论中，得到修正的弗里德曼方程

$$\frac{\ddot{a}}{a} + H^2 - \frac{2}{3}\Lambda = \frac{c}{M_{\rm P}^2}\left[\frac{\dddot{a}}{a} + 3H\frac{\ddot{a}}{a} + \left(\frac{\ddot{a}}{a}\right)^2 - \left(5 + \frac{4b}{c}\right)H^2\frac{\ddot{a}}{a}\right], \tag{98}$$

in which we defined the Planck mass $M_{\rm P}^{-2} = 8\pi G_{\rm N}$. As could have been anticipated, this solution depends on b and c , but neither on ω and a_1 since the Weyl tensor is conformally invariant, nor on a_2 and a_3 (surface terms), and we have set $a_4 \to 0$ to ensure the original theory is conformally invariant.

其中我们定义了普朗克质量 $M_{\mathrm{P}}^{-2} = 8\pi G_{\mathrm{N}}$ 。正如预期，该解依赖于 b 和 c ，但既不依赖于 ω 和 a_1 (因为外尔张量是共形不变的)，也不依赖于 a_2 和 a_3 (表面项)，并且我们设定了 $a_4 \to 0$ 以保证原理论满足共形不变性。

Inflationary solutions for Eq. (98) can be found in Refs. [56, 61-64]. A simple case consists of a de Sitter solution $a \propto \exp(Ht)$ with H constant, which transforms Eq. (98) into a quadratic algebraic equation for H whose solutions

方程 (98) 的暴胀解可见文献 [56, 61-64]。一个简单情形是德西特解 $a \propto \exp(Ht)$ ，其中 H 为常数，这将方程 (98) 转化为关于 H 的二次代数方程，其解

$$H^2 = \frac{M_{\mathrm{P}}^2}{2\,|b|}\left(1 \pm \sqrt{1 + \frac{4\,|b|\,\Lambda}{3M_{\mathrm{P}}^2}}\right) \xrightarrow[|b|\Lambda \ll M_{\mathrm{P}}^2]{} \begin{cases} H_{\mathrm{inf}}^2 = M_{\mathrm{P}}^2/\,|b| & (+) \\ H_\Lambda^2 = 2\Lambda/3 & (-) \end{cases} \tag{99}$$

produce the two relevant extreme cases of present-day cosmological constant domination and initial inflation, with $H_{\mathrm{inf}} \gg H_\Lambda$.

给出了两个相关的极端情形: 当前宇宙学常数主导与初始暴胀，其中 $H_{\mathrm{inf}} \gg H_\Lambda$ 。

Tensor perturbations of the kind (1) in this context are slightly different from those of ordinary GR discussed in the previous sections. In particular, the mode equation (4) is now replaced by the slightly more involved fourth-order equation (see Ref. [65] for details)

本文背景下形式如 (1) 的张量扰动与前文讨论的普通广义相对论中的张量扰动略有不同。具体来说，原模式方程 (4) 现在被替换为形式稍复杂的四阶方程 (细节见文献 [65])

$$\begin{aligned}
&\left(2f_1 + \frac{f_2}{2}\right)\ddddot{ii} + \left[3H\,(4f_1 + f_2) + 4\dot{f}_1 + \dot{f}_2\right]\dddot{i} + \Bigg[3H^2\left(6f_1 + \frac{f_2}{2} - 4f_3\right) \\
&+H\left(16\dot{f}_1 + \frac{9}{2}\dot{f}_2\right) + 6\dot{H}\,(f_1 - f_3) - \frac{16\pi^2}{3}\,|b|\,(H^2 - \dot{H})]\ddot{h} \\
&-(4f_1 + f_2)\,\frac{\nabla^2\ddot{h}}{a^2} + \Bigg[2\dot{H}\left(2\dot{f}_1 - 3\dot{f}_3\right) - \frac{21}{2}H\dot{H}\,(f_2 + 4f_3) - \frac{3}{2}\ddot{H}\,(f_2 + 4f_3) \\
&+3H^2\left(4\dot{f}_1 + \frac{1}{2}\dot{f}_2 - 4\dot{f}_3\right) - 9H^3\,(f_2 + 4f_3) + H\left(4\ddot{f}_1 + \frac{3}{2}\ddot{f}_2 + \frac{3M_{\mathrm{P}}^2}{4}\right) \\
&+\frac{16\pi^2}{3}\,|b|\left(\ddot{H} + H\dot{H} - 3H^3\right)\Bigg]\dot{h} - \left[H\,(4f_1 + f_2) + 4\dot{f}_1 + \dot{f}_2\right]\frac{\nabla^2\dot{h}}{a^2} \\
&+\Bigg[\frac{16\pi^2}{3}\,|b|\left(2\dddot{H} + 12H\ddot{H} + 9\dot{H}^2 - 6H^2\dot{H} - 15H^4\right) + \frac{M_{\mathrm{P}}^2}{2}\left(2\dot{H} + 3H^2\right) \\
&-4H\dot{H}\left(8\dot{f}_1 + 9\dot{f}_2 + 30\dot{f}_3\right) - 8\ddot{H}\left(\dot{f}_1 + \dot{f}_2 + 3\dot{f}_3\right) - H^2\left(4\ddot{f}_1 + 6\ddot{f}_2 + 24\ddot{f}_3\right) \\
&-4\dot{H}\left(\ddot{f}_1 + \ddot{f}_2 + 3\ddot{f}_3\right) - H^3\left(8\dot{f}_1 + 12\dot{f}_2 + 48\dot{f}_3\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(36\dot{H}H^2+18\dot{H}^2+24H\ddot{H}+4\dddot{H}\right)(f_1+f_2+3f_3)\Big]\,h\\
&+\Bigg[2\left(2H^2+\dot{H}\right)(f_1+f_2+3f_3)+\frac{1}{2}H\left(4\dot{f}_1+\dot{f}_2\right)+\frac{M_{\mathrm{P}}^2}{2}-\frac{1}{2}\ddot{f}_2\\
&-\frac{16\pi^2}{3}|b|\left(\dot{H}+5H^2\right)\Bigg]\frac{\nabla^2 h}{a^2}+\left(2f_1+\frac{1}{2}f_2\right)\frac{\nabla^4 h}{a^4}=0,
\end{aligned}
\tag{100}
$$

stemming from the variation of the second-order Lagrangian function

它由二阶拉格朗日函数变分得到

$$
\mathcal{L}=\frac{M_{\mathrm{P}}^2}{2}R+f_1R^2_{\alpha\beta\mu\nu}+f_2R^2_{\alpha\beta}+f_3R^2-\frac{4\pi}{3}\sqrt{|b|}\sigma\Box R+\frac{1}{2}\sigma\Delta\sigma,
\tag{101}
$$

and we have set $\rho=\rho_{\mathrm{h}}\to 0$ and $\sigma_{\mathrm{h}}\to 0$ as the background depends only on time; the perturbation $h(\mathbf{x},t)$ is the amplitude of the tensor mode h_{ij} for a given polarisation. In Eqs. (100) and (101), the coefficients f_1, f_2 and f_3 are time-dependent functions that take the values

由于背景仅依赖于时间，我们设定 $\rho=\rho_{\mathrm{h}}\to 0$ 和 $\sigma_{\mathrm{h}}\to 0$；对给定偏振，扰动 $h(\mathbf{x},t)$ 是张量模式 h_{ij} 的振幅。在方程 (100) 和 (101) 中，系数 f_1, f_2 和 f_3 是依赖于时间的函数，取值为

$$
\begin{aligned}
f_1&=a_1+a_2+\frac{|b|-\omega}{2\sqrt{|b|}}\sigma\\
f_2&=-2a_1-4a_2+\frac{\omega-2|b|}{\sqrt{|b|}}\sigma\\
f_3&=\frac{a_1}{3}+a_2-\frac{3c-2|b|}{36}+\frac{3|b|-\omega}{6\sqrt{|b|}}\sigma.
\end{aligned}
$$

By inspection of the combinations of f ' s entering Eq. (100), one notes that the equation of motion does not depend on a_2 , as expected from the fact that this comes from a surface term.

通过观察进入方程 (100) 的 f 组合，不难发现运动方程不依赖于 a_2 ，这符合预期，因为该项来源于表面项。

Equation (100) was obtained by assuming the value (97) for the auxiliary field $\sigma(t)$ in terms of the background Hubble variable, and so can be used for any admissible solution for the scale factor, including the inflating case of solutions (99). Expanding in Fourier modes, i.e., replacing ∇ by $-\mathbf{k}^2$, in principle permits to evaluate the gravitational wave stochastic spectrum in such a theory, with a catch: contrary to GR, the mode equation is no longer that of a parametric oscillator, so that its quantisation, and consequently the vacuum initial conditions, is not that well defined.

方程 (100) 是在假设辅助场 $\sigma(t)$ 取背景哈勃变量表示的 (97) 式取值后得到的，因此它适用于标度因子的任意容许解，包括解 (99) 的暴胀情形。通过傅里叶模展开，即将 ∇ 替换为 $-\mathbf{k}^2$ ，原则上可以计算该理论下引力波的随机谱，但需要注意: 与广义相对论不同，该模式方程不再是参量谐振子方程，因此它的量子化以及相应的真空初始条件并没有很好的定义。

This issue, still under discussion, can be handled by assuming that our semiclassical framework provides a perturbation to GR, so that the extra (higher derivative) terms may be neglected while quantising in a regime in which one can manage to construct a consistent Hilbert space of state. Setting quantum vacuum fluctuation initial conditions exactly then allows setting initial values for the gravitational wave amplitude and its first three time derivatives.

该问题目前仍在讨论中，可以通过我们的半经典框架对广义相对论提供微扰来处理: 因此在能够构建一致状态希尔伯特空间的区域中量子化时，可以额外的高阶导数项可以忽略。精确设定量子真空涨落初始条件后，即可确定引力波振幅及其前三阶时间导数的初始值。

Moreover, the presence of the higher derivative terms potentially implies instabilities. Setting initial conditions as discussed above, one finds [66,67] that the time development, and hence the resulting predictions, is very sensitive to the properties of the background. Assuming, for instance, a de Sitter inflation phase with constant Hubble rate $H = H_{\text{inf}}$, initial trans-Planckian runaway solutions can be redshifted to become sub-Planckian and then rapidly damped by the expansion: the instabilities indeed present in the theory can end up harmless in a cosmological setup. We assume in what follows that this is indeed the case.

此外, 高阶导数项的存在可能会引发不稳定性。通过采用上文讨论的方式设定初始条件, 文献 [66,67] 发现，时间演化以及由此得到的预言对背景性质十分敏感。例如，假设处于哈勃率恒定为 $H = H_{\text{inf}}$ 的德西特暴胀阶段，初始的跨普朗克快涨解会被红移，成为亚普朗克解，随后被膨胀迅速阻尼: 该理论中确实存在的不稳定性，最终在宇宙学框架中可以是无害的。在下文中我们假设实际情况确实如此。

Primordial Gravitational Wave Background

原初引力波背景

Independently of the underlying quantum theory leading to the production of primordial tensor modes, one must now evolve them through the expanding universe to evaluate their current contribution. As we know GR to be valid for the most part of the FLRW evolution, we consider from now on that the higher derivative terms discussed above either are not present at all or contribute only negligibly. In order to clearly distinguish classical from quantum effects, we include again the relevant factors of $\hbar$ when necessary.

无论导致原初张量模式产生的基础量子理论是什么，我们现在都需要让这些模式在膨胀宇宙中演化，以计算它们当前的贡献。由于我们已知广义相对论在 FLRW 演化的大部分过程中都是成立的，因此从现在开始我们认为，前文讨论的高阶导数项要么根本不存在，要么贡献可以忽略不计。为了清晰区分经典效应和量子效应，必要时我们会再次引入 $\hbar$ 的相关因子。

In section "Particle Production", we have laid out three equivalent ways to describe the evolution of perturbations for a general time-dependent background $a(\eta)$: the use of Bogoliubov transformations, mode functions and squeezing parameters. We now solve the dynamics of the gravitational wave field in a simplified model of the cosmological evolution to discuss the properties of the primordial gravitational waves generated and make a connection with observations.

在“粒子产生”一节中，我们给出了描述一般含时背景 $a(\eta)$ 下微扰演化的三种等价方法：玻戈留波夫变换、模函数和压缩参数。我们现在将在一个简化的宇宙学演化模型中求解引力波场的动力学，以讨论所产生的原初引力波的性质，并建立其与观测的联系。

Cosmological Evolution

宇宙演化

In FLRW the curvature of spacetime is contained in the scale factor a , whose dynamics is related to the matter content of the Universe through the Friedmann equations. In what follows, we first solve them in the standard approximation that there is always a single fluid dominating the energy budget of the Universe and that transitions between two phases are instantaneous. One can thus model the cosmological evolution as a succession of three eras: first an accelerated expansion phase for $-\infty \le \eta \le \eta_{\mathrm{r}}$, whose dynamics is that of a slow-roll inflation phase [68], then a radiation dominated phase for $\eta_{\mathrm{r}} \le \eta \le \eta_{\mathrm{m}}$ and finally a matter domination for $\eta \ge \eta_{\mathrm{m}}$. For the sake of simplicity, we ignore the late-time accelerated expansion.

在 FLRW 宇宙中，时空曲率包含在标度因子 a 中，其动力学通过弗里德曼方程与宇宙的物质含量相关联。下文我们将在标准近似下求解这些方程，该近似假设始终只有单一流体主导宇宙的能量收支，且两个阶段之间的转变是瞬时的。因此可以将宇宙演化建模为三个连续的时代：首先是 $-\infty \le \eta \le \eta_{\mathrm{r}}$ 的加速膨胀阶段，其动力学对应慢滚暴胀阶段 [68]，随后是 $\eta_{\mathrm{r}} \le \eta \le \eta_{\mathrm{m}}$ 的辐射主导阶段，最后是 $\eta \ge \eta_{\mathrm{m}}$ 的物质主导阶段。为简化起见，我们忽略后期加速膨胀。

The evolution of the gravitational waves contained in the universe is controlled by Eq. (4) where the expansion enters through the scale factor $a(\eta)$ and its second derivative. Connecting the scale factor and its derivative continuously across the transitions, we have

宇宙中引力波的演化由式 (4) 控制，其中膨胀通过标度因子 $a(\eta)$ 及其二阶导数引入。在各阶段转变之间连续连接标度因子及其导数，我们得到

$$\frac{a(\eta)}{a_{\mathrm{r}}} = \begin{cases} \frac{\eta_{\mathrm{r}}^{1+\varepsilon}}{(2\eta_{\mathrm{r}}-\eta)^{1+\varepsilon}} \approx \frac{\eta_{\mathrm{r}}}{2\eta_{\mathrm{r}}-\eta} + \mathcal{O}(\varepsilon) & \text{for} \quad -\infty \le \eta \le \eta_{\mathrm{r}}, \\ \frac{\eta}{\eta_{\mathrm{r}}} & \text{for} \quad \eta_{\mathrm{r}} \le \eta \le \eta_{\mathrm{m}}, \\ \frac{\eta_{\mathrm{m}}}{2\eta_{\mathrm{r}}}\left(\frac{\eta^2}{\eta_{\mathrm{m}}^2}+1\right) & \text{for} \quad \eta_{\mathrm{m}} \le \eta, \end{cases} \tag{102}$$

where $\eta_{\mathrm{r}} > 0$. The first expression in inflation is at first order in $\varepsilon = 1 - \mathcal{H}'/\mathcal{H}^2$, the first slow-roll parameter considered time-independent, and we have also given the de Sitter limit $\varepsilon = 0$. From this, one computes the time-dependent part of the frequency ω_k^2 defined in Eq. (5)

其中 $\eta_{\mathrm{r}} > 0$ 。暴胀阶段的第一个表达式是 $\varepsilon = 1 - \mathcal{H}'/\mathcal{H}^2$ 的一阶结果，$\varepsilon = 1 - \mathcal{H}'/\mathcal{H}^2$ 是被视为时间无关的第一个慢滚参数，我们同时给出了德西特极限 $\varepsilon = 0$ 。由此可以计算出式 (5) 定义的频率 ω_k^2 的含时部分

$$\frac{a''}{a} = \begin{cases} \frac{2+3\varepsilon}{(2\eta_{\rm r}-\eta)^2} \approx \frac{2}{(\eta_{\rm r}-\eta)^2} + \mathcal{O}(\varepsilon) & \text{for} \quad -\infty \leq \eta \leq \eta_{\rm r}, \\ 0 & \text{for} \quad \eta_{\rm r} \leq \eta \leq \eta_{\rm m}, \\ \frac{2}{\eta^2} & \text{for} \quad \eta_{\rm m} \leq \eta. \end{cases} \tag{103}$$

Solving Eq. (4) with evolution (103) yields reference mode functions in each era, namely,

结合演化关系 (103) 求解式 (4)，得到每个时代的参考模函数，即:

$$u_k^{(\rm infl.)}(\eta) = \sqrt{\frac{-(\eta-2\eta_{\rm r})\hbar\pi}{4}} H^{(1)}_{\frac{3}{2}+\varepsilon}\left[-k(\eta-2\eta_{\rm r})\right] \quad \text{for} \ -\infty \leq \eta \leq \eta_{\rm r},$$

(104a)

$$\approx \sqrt{\frac{\hbar}{2k}} e^{-ik(\eta-2\eta_{\rm r})}\left[1-\frac{i}{k(\eta-2\eta_{\rm r})}\right]$$

$$u_k^{(\rm r)}(\eta) = \sqrt{\frac{\hbar}{2k}} e^{-ik\eta} = u_k^{(\rm M)}(\eta) \quad \text{for} \ \ \eta_{\rm r} \leq \eta \leq \eta_{\rm m},$$

(104b)

$$u_k^{(\rm m)}(\eta) = \sqrt{\frac{\hbar}{2k}} e^{-ik\eta}\left(1-\frac{i}{k\eta}\right) \tag{104c}$$

where in the first line, $H_\kappa^{(1)}$ is the Hankel function of the first kind of index κ and the approximation corresponds to the de Sitter limit. We refer to Ref. [69] for a recent textbook in which all details of the computations of the inflationary mode function can be found. Note that during radiation domination, the solution is given by the Minkowski mode function because $a'' = 0$. Since two solutions of (4) are related by a Bogoliubov transformation, a mode function solution of (4) for the whole cosmological evolution is related by a Bogoliubov transformation to the associated reference mode function (104) in each era.

其中第一行中，$H_\kappa^{(1)}$ 是指标为 κ 的第一类汉克尔函数，对应近似为德西特极限。关于暴胀模函数计算的所有细节，读者可参考最新教材文献 [69]。注意，辐射主导时期，解由闵可夫斯基模函数给出，这是因为 $a'' = 0$ 。由于式 (4) 的任意两个解通过博戈留波夫变换关联，因此整个宇宙演化过程中式 (4) 的模函数解，与每个时代对应的参考模函数 (104) 同样通过博戈留波夫变换关联。

One can construct a global solution $u_k(\eta)$ starting in the inflationary period. The reference mode function there was chosen to match the Minkowski mode function $u_k^{(\rm M)}$ in the asymptotic past $\eta \to -\infty$. This gives us an "in" region in which we can set the initial condition for the state of the system in terms of a well-defined particle content. We therefore pick $u_k(\eta) = u_k^{(\rm M)}(\eta)$ during inflation. The expressions for the radiation and matter domination are then

我们可以构造从暴胀时期开始的全局解 $u_k(\eta)$ 。该处选择参考模函数是为了匹配渐近过去 $\eta \to -\infty$ 处的闵可夫斯基模函数 $u_k^{(\rm M)}$ 。这给我们提供了一个“入”区域，我们可以在该区域基于定义明确的粒子含量设置系统状态的初始条件。因此我们在暴胀阶段选择 $u_k(\eta) = u_k^{(\rm M)}(\eta)$ 。辐射主导和物质主导阶段的表达式则为

$$u_k(\eta) = \begin{cases} \alpha_k^{(\rm r)} u_k^{(\rm r)}(\eta) + \beta_k^{(\rm r)} u_k^{\star(\rm r)}(\eta) & \text{for} \quad \eta_{\rm r} \leq \eta \leq \eta_1 \\ \alpha_k^{(\rm m)} u_k^{(\rm m)}(\eta) + \beta_k^{(\rm m)} u_k^{\star(\rm m)}(\eta) & \text{for} \quad \eta_{\rm m} \leq \eta, \end{cases} \tag{105}$$

where the Bogoliubov coefficients are found by requiring that the mode function and its first-time derivative are continuous across the transition. Their expressions are worked out in full in Ref. [45]. The mode $u_k(\eta)$ is then completely determined for both polarisations, and, using Eq. (38), one achieves a fully quantum description of the evolution of the gravitational wave field.

> 其中博戈留波夫系数通过要求模函数及其一阶时间导数在阶段转变处连续得到，其完整推导可见文献 [45]。至此，两种偏振的模 $u_k(\eta)$ 都被完全确定，结合式 (38)，我们就得到了引力波场演化的完全量子描述。

The analysis is completed once one specifies the initial state of the gravitational waves as $k\eta \to -\infty$. The standard choice is to assume that, in the far past, the inflation phase somehow wiped out any initial perturbation, leaving no graviton to start with: this is the motivation behind choosing the vacuum state for every mode. This vacuum initial state is often referred to the Bunch-Davies vacuum [25], although it should be more appropriately be called Minkowski vacuum. This choice implies that the state of the perturbation consists of a collection of independent two-mode squeezed states as discussed in section ”Squeezed States”.

> 只要指定引力波的初始态为 $k\eta \to -\infty$ ，分析就完成了。标准选择是假设在极远过去，暴胀阶段已经抹除了所有初始扰动，初始不存在引力子: 这就是每个模式都选择真空态的动机。这种真空初始态通常被称为邦奇-戴维斯真空 [25]，更恰当的称呼应该是闵可夫斯基真空。正如“压缩态”一节所讨论的，这个选择意味着扰动态是一系列独立双模压缩态的集合。

For scalar perturbations, the above vacuum choice turns out to be in excellent agreement with the observations of the cosmic microwave background [70]. For gravitational waves, we are so far short of equivalent observations so that other states could be chosen as initial condition [71]. Although such alternative choices do not modify our description of the subsequent evolution, they change the values of the Bogoliubov coefficients and therefore the prediction on the amplitude of gravitational waves or, equivalently, the number of gravitons produced.

> 对于标量微扰而言，上述真空选择与宇宙微波背景的观测结果 [70] 极为吻合。对于引力波，我们目前还缺乏同等精度的观测，因此可以选择其他状态作为初始条件 [71]。尽管这些替代选择不会改变我们对后续演化的描述，但它们会改变博戈留波夫系数的值，从而改变对引力波振幅 (或者 equivalently，产生的引力子数量) 的预言。

We have explained in section ”Particle Production” that, most of the time, this number is ambiguous due to the time-dependent part of ω_k^2 . Let us explain how to make sense of it for primordial gravitational waves. First, in the sub-Hubble regime $k^2 \gg a''/a$, the frequency reduces to $\omega_k \sim k$, i.e., the mode $\mathbf{k}$ does not feel the expansion of space and effectively oscillates as in flat spacetime. In this sub-Hubble limit, the reference mode functions (104) reduce to the Minkowski one, and we can treat the mode as if evolving in Minkowski. On the other hand, in the super-Hubble regime $k^2 \ll a''/a$, the mode behaves as an inverted harmonic oscillator $\omega_k \sim -a''/a < 0$. One therefore expects its amplitude to be amplified, and it is indeed where most of the squeezing happens, as illustrated in the first two panels of Fig. 5.

我们在“粒子产生”章节已经说明，多数情况下，由于 ω_k^2 含时部分的存在，这个数量是不明确的。下面我们说明对于原初引力波该如何理解这个概念。首先，在亚哈勃区域 $k^2 \gg a''/a$ 中，频率约化为 $\omega_k \sim k$ ，即模式 $\mathbf{k}$ 感受不到空间膨胀，其振荡行为和平直时空一致。在该亚哈勃极限下，参考模函数 (104) 退化为闵氏模函数，我们可以将该模式当作在闵氏时空中演化来处理。另一方面，在超哈勃区域 $k^2 \ll a''/a$ ，模式的行为等价于反谐振子 $\omega_k \sim -a''/a < 0$ 。因此可以预期其振幅会被放大，正如正如图 5 前两张子图所展示的，大部分压缩效应确实就发生在这里。

The evolution (103) of the time-dependent piece a''/a is plotted in Fig. 4 and is compared to the square of the comoving frequencies of two different modes k_s^2 and k_1^2 . Note that at the beginning of inflation and during the radiation era, since $a'' = 0$, all modes are sub-Hubble and effectively living in Minkowski there. Recall that in this limiting case, the relation between the dominant term in the frequency and the wavelength size compared to the Hubble radius does not hold. One cannot, strictly speaking, employ the terminology sub- or super-Hubble here. This second aspect is due to our simplistic modelling of the transition in Eq. (102). In a realistic cosmological model, a'' is continuous, and part of the modes progressively reach the sub-Hubble regime. In Fig. 4, the mode $\mathbf{k}_s$ has a short wavelength and is always sub-Hubble. It is not affected by the amplification process. The mode $\mathbf{k}_1$ has a larger wavelength and becomes super-Hubble during inflation after $\eta_{k,1}$, is insensitive to the expansion during radiation domination and becomes super-Hubble again during matter domination, until $\eta_{k,2}$ where it settles in the sub-Hubble regime. The modes of interest for cosmological observations are of the second type (or become and stay super-Hubble during radiation domination).

含时部分 a''/a 的演化 (103) 绘制在图 4 中，同时与两个不同模式 k_s^2 和 k_1^2 的共动频率平方做了对比。注意在暴胀初期和辐射时代，由于 $a'' = 0$ ，所有模式都是亚哈勃模式，等效于处于闵氏时空中。需要回顾的是，在该极限情况下，频率主导项和波长与哈勃半径相对大小的对应关系并不成立。严格来说，我们不能在此使用亚哈勃或超哈勃的术语。这一问题来源于我们对式 (102) 中跃迁过程的简化建模。在 realistic 宇宙学模型中，a'' 是连续的，部分模式会逐步进入亚哈勃区域。在图 4 中，模式 $\mathbf{k}_s$ 波长较短，始终处于亚哈勃尺度，不会受到放大过程的影响。模式 $\mathbf{k}_1$ 波长更长，在 $\eta_{k,1}$ 之后的暴胀阶段变为超哈勃模式，在辐射主导阶段对膨胀不敏感，又在物质主导阶段再次变为超哈勃模式，直到 $\eta_{k,2}$ 才稳定进入亚哈勃区域。宇宙学观测感兴趣的是第二类模式 (或是在辐射主导阶段变为并保持超哈勃的模式)。

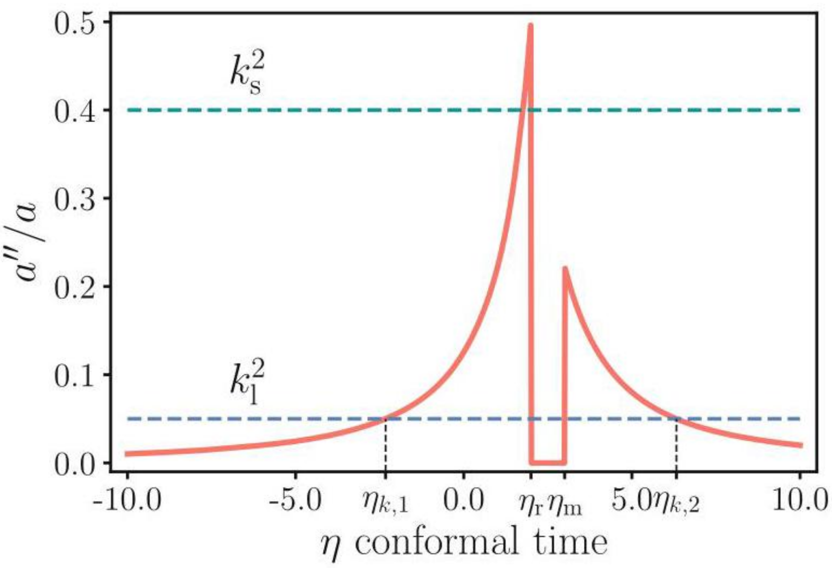


Fig. 4 Sketch of the potential for the tensor mode within the toy model (103). The full red line represents the evolution of a''/a in arbitrary units in our simplified cosmological evolution (102). The blue and green dotted lines represent two comoving frequencies k_{s}^2 and k_1^2 in arbitrary units which are constant during the evolution

图 4 玩具模型 (103) 中张量模的势示意图。红色实线代表我们简化宇宙演化 (102) 中 a''/a 以任意单位绘制的演化曲线。蓝色和绿色虚线代表两个共动频率 k_{s}^2 和 k_1^2 以任意单位绘制的曲线，二者在演化过程中保持不变

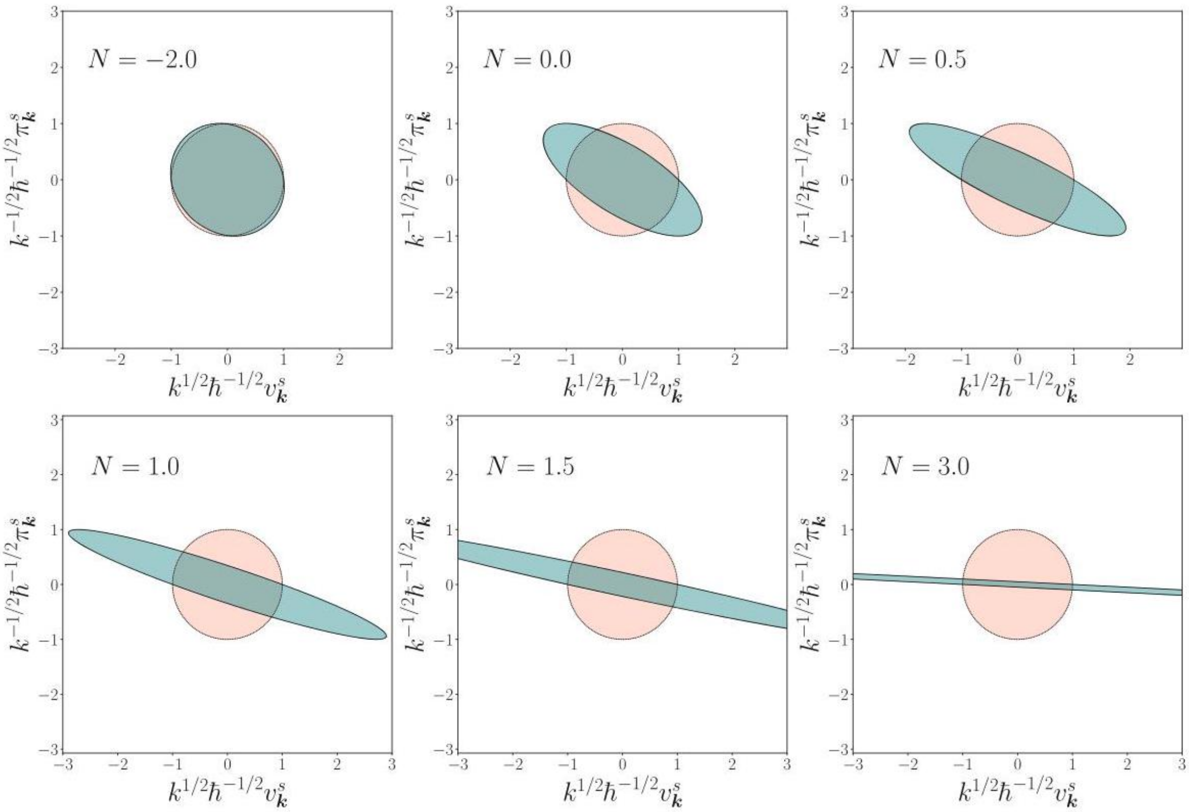


Fig. 5 Phase space ellipse in the plane $\left(k^{1/2}\mu_k^{\mathrm{S}}, k^{-1/2}\pi_k^{\mathrm{S}}\right)$ at different instants during inflation, labelled by $N = \ln\left[a/a\left(\eta_k\right)\right]$, i.e., the number of e -folds measured from the Hubble-crossing time of the mode under consideration. On sub-Hubble scales, the ellipse remains roughly a circle, while it gets squeezed and rotates in the super-Hubble regime

图 5 暴胀过程不同时刻平面 $\left(k^{1/2}\mu_k^{\mathrm{S}}, k^{-1/2}\pi_k^{\mathrm{S}}\right)$ 上的相空间椭圆，时刻由 $N = \ln\left[a/a\left(\eta_k\right)\right]$ 标记，即 e 折叠数从所研究模式的哈勃穿越时刻开始计算。在亚哈勃尺度上，椭圆大致保持圆形，进入超哈勃区域后会被压缩并发生旋转

The picture that we have just sketched for these modes, putting aside radiation domination, is reminiscent of the idealized situation described in section "Particle Production" where the "in" region corresponds to $\eta \ll \eta_{k,1}$ and the "out" region to $\eta \gg \eta_{k,2}$. For such modes, we are thus justified in talking about graviton production.

我们刚刚勾勒出的这些模式的图像 (暂不考虑辐射主导时期)，让人想起“粒子产生”一节中描述的理想情景: 其中“入”区域对应 $\eta \ll \eta_{k,1}$ ，“出”区域对应 $\eta \gg \eta_{k,2}$ 。因此，我们有理由讨论这些模式的引力子产生。

Two remarks are in order here. First, modes progressively reenter the Hubble radius during the neglected current accelerated expansion. For the modes of interest here, this is of no consequence. Second, one should be careful when discussing modes responsible for the **B** -modes of polarisation in the CMB since some of them had not yet reached the sub-Hubble regime when generating polarisation.

这里有两点需要说明。第一，在我们忽略的当前加速膨胀阶段，模式会逐渐重新进入哈勃半径，这对本文讨论的感兴趣的模式没有影响。第二，讨论对 CMB 中 **B** 偏振模式有贡献的模式时需要注意，在产生偏振的时候，其中一些模式还没有进入哈勃亚尺度区域。

To close this discussion, we compute the relevant quantities describing the gravitons in the different formalisms. For simplicity, we only consider an inflationary period where most of the amplification occurs. For this estimate, we neglect slow-roll corrections and model inflation by a period of de Sitter expansion ending at η_{r} . After the transition to radiation domination, the mode does not feel the expansion anymore, so that its particle content can be computed. In de Sitter, the covariance matrix elements can be computed exactly using the mode function in Eqs. (104). We evaluate them at η_{r} [37]

为结束本次讨论，我们在不同形式体系下计算描述引力子的相关物理量。为简便起见，我们仅考虑发生大部分放大效应的暴胀阶段。做这个估计时，我们忽略慢滚修正，将暴胀建模为结束于 η_{r} 的德西特膨胀阶段。在过渡到辐射主导后，模式不再受膨胀影响，因此可以计算其粒子数。在德西特空间中，可以利用式 (104) 的模式函数精确计算协变矩阵元，我们在 η_{r} 处对其求值 [37]

$$\gamma_{11} = 1 + \frac{1}{k^2\eta_{\mathrm{r}}^2} \approx e^{2N},\ \gamma_{12} = -\frac{1}{k\eta_{\mathrm{r}}} \approx e^{N},\ \gamma_{22} = 1. \tag{106}$$

where, since we are considering a mode which is in the super-Hubble regime during inflation, we have taken the limit $k\eta_{\mathrm{r}} \ll 1$. These last expressions are given in terms of the number of e -folds N defined by $N = \ln\left[a(\eta)/a(\eta_k)\right] = \ln\left[k(2\eta_{\mathrm{r}} - \eta)\right]$ where $a(\eta_k)$ is the scale factor evaluated at Hubble crossing time $k(2\eta_{\mathrm{r}} - \eta_k) = 1$.

由于我们考虑的是暴胀期间处于哈勃超尺度区域的模式，这里我们取了极限 $k\eta_{\mathrm{r}} \ll 1$ 。最后这些表达式用 e 折叠数 N 表示，N 定义为 $N = \ln\left[a(\eta)/a(\eta_k)\right] = \ln\left[k(2\eta_{\mathrm{r}} - \eta)\right]$ ，其中 $a(\eta_k)$ 是哈勃穿越时刻 $k(2\eta_{\mathrm{r}} - \eta_k) = 1$ 处的尺度因子。

At this point, one notes that $\left\langle\left(\hat{h}'_{\lambda,\mathbf{k}}\hat{h}'_{\lambda,-\mathbf{k}}\right)^2\right\rangle \propto \gamma_{22}/a^2(\eta)$, so that, for a super-Hubble mode, it decays exponentially during inflation. The fact that the matrix element $\gamma_{11} = 2k\left\langle\left(\hat{\mu}_k^{\mathrm{R}}\right)^2\right\rangle$ grows faster than γ_{12} and γ_{22} leads to squeezing in a direction close to that of the μ_k^{R} axis. This can be verified by computing explicitly the squeezing parameters: inverting Eq. (66), we deduce the squeezing parameters as

此时我们注意到 $\left\langle\left(\widehat{h}'_{\lambda,\mathbf{k}}\widehat{h}'_{\lambda,-\mathbf{k}}\right)^2\right\rangle \propto \gamma_{22}/a^2(\eta)$，因此对于哈勃超尺度模式，它在暴胀期间指数衰减。矩阵元 $\gamma_{11} = 2k\left\langle\left(\widehat{\mu}_k^{\mathrm{R}}\right)^2\right\rangle$ 比 γ_{12} 和 γ_{22} 增长更快，这导致压缩方向接近 μ_k^{R} 轴。这可以通过直接计算压缩参数验证: 反转式 (66)，我们推导出压缩参数为

$$r_k = \operatorname{arcsinh}\left(\frac{1}{2k\eta_{\mathrm{r}}}\right), \ \varphi_k = \frac{\pi}{2} - \frac{1}{2}\arctan\left(2k\eta_{\mathrm{r}}\right), \tag{107}$$

which, in the super-Hubble limit, yields

在哈勃超极限下，它给出

$$r_k \approx \ln\left(\frac{1}{2k\eta_{\mathrm{r}}}\right) \approx N, \ \varphi_k \approx \frac{\pi}{2} - k\eta_{\mathrm{r}} \approx \frac{\pi}{2} - e^{-N}. \tag{108}$$

Since $\varphi_k \to \pi/2$ then indeed the ellipse will be squeezed in a direction close to μ_k^{R} .

由于 $\varphi_k \to \pi/2$，因此椭圆确实会在接近 μ_k^{R} 的方向上被压缩。

For scales of cosmological interest, one typically expects $N = \ln(k\eta_{\mathrm{r}}) \sim 50$ at the end of inflation so that $r \approx 50$. This is to be compared with the best quantum optics experiments where one can hardly achieve $r \approx 2$; the squeezing is extreme [72]. The resulting evolution of the Wigner function is plotted for a few e -folds after Hubble exit in Fig. 5 where the very large squeezing is manifest.

对于宇宙学感兴趣的尺度，我们通常认为暴胀结束时 $N = \ln(k\eta_{\mathrm{r}}) \sim 50$，因此 $r \approx 50$。对比来看，最优量子光学实验也几乎达不到 $r \approx 2$；这里的压缩程度极高 [72]。图 5 绘制了哈勃出射后几个 e 折叠内维格纳函数的演化，其中非常大的压缩程度非常明显。

Finally, we compute the number of particles created and their pair correlation: Eq. (78) in the de Sitter and super-Hubble limits gives

最后，我们计算产生的粒子数及其对关联: 在德西特极限和哈勃超极限下，式 (78) 给出

$$n_k = \frac{1}{4k^2\eta^2} = \frac{e^{2N}}{4}, \tag{109}$$

$$c_k = \frac{1}{4k^2\eta^2} - \frac{i}{2k\eta} \approx \frac{e^{2N}}{4}. \tag{110}$$

The number of pairs and their correlation grow at the same rate; squeezing necessarily creates entangled pairs. After $50e$ -folds of inflation, one finds $n_k \sim 10^{43}$. This number might appear very large, but the physical field h_{ij} is diluted by the inverse of the scale factor that will keep acting even when the creation process stops, following Eq. (3). In addition, the number of gravitons is not directly observable; we observe gravitational waves or their imprint on other fields, e.g., the electromagnetic field in the CMB, but not individual gravitons. One therefore needs to compute the physical quantities that are more directly relevant in forecasting future observations.

对的对数及其关联以相同速率增长；压缩过程必然会产生纠缠对。经过 50e 次暴胀折叠后，可得到 $n_k \sim 10^{43}$ 。这个数值看起来很大，但根据式 (3)，即使产生过程停止，物理场 h_{ij} 仍会被尺度因子的倒数持续稀释。此外，引力子的数量无法直接观测；我们观测到的是引力波或它们在其他场中的印记，例如宇宙微波背景中的电磁场，而非单个引力子。因此需要计算对预测未来观测更直接相关的物理量。

Connection to Observations

与观测的联系

There is hope that observable signatures of these primordial gravitational waves will be found either in the **B** -modes of the CMB or directly in future gravitational wave interferometers. We refer to Ref. [73] or Chapters 19, 20, and 23 in Ref. [33] for a detailed account. The waves we have described are stochastic in nature owing to their quantum origin. They account for part of the stochastic gravitational wave background (SGWB), the rest being produced by unresolved astrophysical sources or possibly other high-energy phenomena such as topological defects. The SGWB is usually assumed to be as statistically homogeneous and isotropic as the FLRW background metric, Gaussian, either due to the sum of a large number of independent sources or because it is sourced by a Gaussian state as considered here, and unpolarised (same content in both polarisations and polarisations are uncorrelated) because there is no significant source of parity violation in the Universe [73]. It can be checked using Eq. (17) that the assumptions that the waves are both unpolarised $\langle \mu_+ \mu_+^\star \rangle = \langle \mu_\times \mu_\times^\star \rangle$ and that polarisations uncorrelated $\langle \mu_+ \mu_\times^\star \rangle = 0$ in terms of the +,- helicity basis is equivalent to the same two assumptions on the $+, \times$ basis. All these assumptions only have to be made on the initial state as the dynamics is the same for both fields $\hat{\mu}_\lambda$ and preserves isotropy and homogeneity. They are in particular satisfied for primordial gravitational waves produced from the Bunch-Davies vacuum. A typical quantity used to characterize a stochastic ensemble of waves is their power spectrum which, within the Gaussianity assumption, contains all the information. The power spectrum $\mathcal{P}_\mathrm{T}$ of gravitational waves h_{ij} at time η is then defined (working classically for the moment) by

我们有望在宇宙微波背景的 **B** 模中发现这些原初引力波的可观测信号，或是在未来的引力波干涉仪中直接探测到它们。详细论述可参见文献 [73] 或文献 [33] 的第 19、20 和 23 章。我们描述的这些引力波源于量子效应，本质上是随机的。它们构成了随机引力波背景 (SGWB) 的一部分，该背景的其余部分来自未解析的天体物理源，也可能来自拓扑缺陷等其他高能现象。由于大量独立源的叠加，或是如本文所讨论由高斯态产生，随机引力波背景通常被认为具有与 FLRW 背景度规相同的统计均匀性和各向同性，且满足高斯性；又因为宇宙中不存在显著的宇称破缺源，它还是非偏振的 (两种偏振分量强度相同，且偏振之间互不相关)[73]。利用式 (17) 可以验证，在 +,-螺旋度基下引力波非偏振 $\langle \mu_+ \mu_+^\star \rangle = \langle \mu_\times \mu_\times^\star \rangle$ 且偏振互不相关 $\langle \mu_+ \mu_\times^\star \rangle = 0$ 的假设，等价于在 +、× 基下的这两个相同假设。所有这些假设仅需要对初态作出，因为两个场 $\hat{\mu}_\lambda$ 的演化动力学完全相同，且动力学过程会保持各向同性和均匀性。对于产生自邦奇-戴维斯真空的原初引力波，这些假设都成立。表征随机引力波系综的一个典型物理量是功率谱，在高斯假设下，功率谱包含了全部信息。我们将时刻 η 处引力波 h_{ij} 的功率谱 $\mathcal{P}_\mathrm{T}$ 定义如下 (目前从经典角度讨论)

$$\langle \mu_\lambda (\mathbf{k}, \eta) \mu_{\lambda'}^\star (\mathbf{k}', \eta) \rangle = \frac{\pi a^2(\eta)}{32 G_\mathrm{N} k^3} \delta^{(3)} (\mathbf{k} - \mathbf{k}') \delta_{\lambda,\lambda'} \mathcal{P}_\mathrm{T}(k, \eta), \tag{111}$$

where the Dirac delta comes from homogeneity and $\mathcal{P}_{\mathrm{T}}$ only depends on k since the background is isotropic and unpolarised. The index "T" stands for "tensor", to differentiate the latter from the scalar power spectrum $\mathcal{P}_{\mathrm{S}}$. Using Eqs. (3),(15) and the orthogonality relations of the tensors below Eq. (16), one finds the two-point correlation function of the Fourier coefficients of h_{ij}

其中狄拉克 δ 函数源于均匀性，且由于背景各向同性、非偏振，$\mathcal{P}_{\mathrm{T}}$ 仅依赖于 k 。下标“T”代表“张量”，用以区分标量功率谱 $\mathcal{P}_{\mathrm{S}}$ 。利用式 (3)、(15) 以及式 (16) 下方张量的正交关系，可以得到 h_{ij} 傅里叶系数的两点关联函数

$$\left\langle h^{ij}(\mathbf{k},\eta)\, h_{ij}^{\star}(\mathbf{k},\eta)\right\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,\frac{2\pi^2}{k^3}\mathcal{P}_{\mathrm{T}}(k,\eta)\,,$$

as well as the two-point correlation of h_{ij} in real space, namely,

以及实空间中 h_{ij} 的两点关联，即

$$\left\langle h^{ij}(\mathbf{x},\eta)\, h_{ij}(\mathbf{x},\eta)\right\rangle = \int \mathrm{d}\ln(k)\,\mathcal{P}_{\mathrm{T}}(k,\eta)\,.$$

The power spectrum $\mathcal{P}_{\mathrm{T}}(k,\eta)$ corresponds to the typical squared amplitude of the wave, per logarithm of k , at the time η . For perturbations made of sub-Hubble modes $k \gg |\mathcal{H}|$, the time-dependent term of (4) can be neglected, and the energy density of gravitational waves reads. Averaging is necessary even for a deterministic source of gravitational waves to make sense of their energy. The averaging can either be performed over a certain volume or a certain duration; see Ref. [33]. In the context of this review, the averaging in Eq. (112) refers to an ensemble average.

功率谱 $\mathcal{P}_{\mathrm{T}}(k,\eta)$ 对应 k 对数单位下，时刻 η 处波的典型振幅平方。对于哈勃半径内模式 $k \gg |\mathcal{H}|$ 构成的微扰，式 (4) 中的时间相关项可以忽略，此时引力波的能量密度可写为。即使对于确定性的引力波源，也需要进行平均才能得到有意义的能量结果。平均可以在一定体积或一定时间范围内进行；参见文献 [33]。在本综述中，式 (112) 中的平均指系综平均。

$$\rho_{\mathrm{GW}} = \frac{1}{32\pi G_{\mathrm{N}}}\left\langle \dot{h}^{ij}(\mathbf{x},\eta)\,\dot{h}_{ij}(\mathbf{x},\eta)\right\rangle. \tag{112}$$

For sub-Hubble modes, $h_{+,\times}(\mathbf{k},\eta) \propto e^{i(\mathbf{k}\cdot\mathbf{x}-k\eta)}/a(\eta)$ so that neglecting terms in $\mathcal{H}$ with respect to k , we get

对于哈勃半径内模式，$h_{+,\times}(\mathbf{k},\eta) \propto e^{i(\mathbf{k}\cdot\mathbf{x}-k\eta)}/a(\eta)$ ，因此相对于 k 忽略 $\mathcal{H}$ 中的项后，我们得到

$$\left\langle \dot{h}^{ij}(\mathbf{k},\eta)\,\dot{h}_{ij}(\mathbf{k},\eta)\right\rangle \approx k^2\frac{\left\langle h^{ij}(\mathbf{k},\eta)\,h_{ij}(\mathbf{k},\eta)\right\rangle}{a^2(\eta)}. \tag{113}$$

Note that, since h_{ij} dilutes as a^{-1}, ρ_{GW} dilutes as a^{-4} , i.e., sub-Hubble modes dilute as standard radiation. Expanding the energy density in Fourier space and normalising by the critical energy density $\rho_{\mathrm{c}} = 3H^2/8\pi G_{\mathrm{N}}$, we get the energy fraction per logarithm of k that is directly expressed as a function of the power spectrum

请注意，由于 h_{ij} 随 a^{-1}，ρ_{GW} 稀释，与 a^{-4} 的稀释规律一致，即哈勃半径内模式的稀释规律与标准辐射相同。将能量密度在傅里叶空间展开，并以临界能量密度 $\rho_{\mathrm{c}} = 3H^2/8\pi G_{\mathrm{N}}$ 归一化后，我们得到每对数单位 k 对应的能量占比，它可以直接表示为功率谱的函数

$$\Omega_{\mathrm{GW}}(k,\eta) = \frac{1}{\rho_{\mathrm{c}}}\frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\ln k} = \frac{k^2}{12H^2a^2(\eta)}\mathcal{P}_{\mathrm{T}}(k,\eta). \tag{114}$$

The power spectrum (111) and the energy density fraction (114) are the two quantities customarily used to assess the observability and constrain the models of primordial gravitational waves. More precisely, we often estimate the primordial power spectrum, i.e., the power spectrum at the beginning of radiation domination. The rest of the evolution is encoded in so-called transfer functions; these can be estimated using the previous computations. For actual comparison with observations, they have to be computed numerically by solving Boltzmann-like equations.

功率谱 (111) 和能量密度占比 (114) 是评估原初引力波可观测性、约束原初引力波模型的两个常用物理量。更准确地说，我们通常估计的是原初功率谱，即辐射主导阶段初期的功率谱，后续演化过程被编码在传递函数中；传递函数可以通过前文的计算得到。若要与实际观测对比，还需要通过求解类玻尔兹曼方程对其进行数值计算。

Let us then evaluate the primordial power spectrum by considering only the initial phase of single-field slow-roll inflation in the cosmological evolution Eq. (102) and assuming Bunch-Davies vacuum for both polarisations $\pm$. Using Eq. (38), the power spectrum is straightforwardly expressed in terms of the mode function for $\hat{\mu}_k$

下面我们来计算原初功率谱，仅考虑宇宙演化方程 (102) 中单场慢滚暴胀的初始阶段，并假设两个偏振态都满足邦奇-戴维斯真空 $\pm$ 。利用式 (38)，可直接将功率谱用 $\hat{\mu}_k$ 的模式函数表示出来

$$\mathcal{P}_{\mathrm{T}}(k,\eta) = \frac{32G_{\mathrm{N}}k^3}{\pi a^2(\eta)}\left|u_k^{(\,\mathrm{infl.}\,)}(\eta)\right|^2. \tag{115}$$

Making use of Eq. (104), we get

利用式 (104)，我们得到

$$\mathcal{P}_{\mathrm{T}}(k,\eta) = 8G_{\mathrm{N}}\frac{H_k^2\hbar}{(1+\varepsilon)^{2+\varepsilon}}\left[-k(\eta-2\eta_{\mathrm{r}})\right]^{3+2\varepsilon}\left|H^{(1)}_{\frac{3}{2}+\varepsilon}\left[-k(\eta-2\eta_{\mathrm{r}})\right]\right|^2,$$

(116)

where η_k is the Hubble crossing time $k/a(\eta_k) = H(\eta_k)$. To be consistent, we expand all the quantities at first order in the slow-roll parameter ε and take the super-Hubble limit relevant for cosmological scales $|k(\eta-2\eta_{\mathrm{r}})| \ll 1$, i.e.,

其中 η_k 是哈勃穿越时刻 $k/a(\eta_k) = H(\eta_k)$ 。为保持自洽，我们将所有物理量对慢滚参数 ε 展开到一阶，并取宇宙学尺度对应的超哈勃极限 $|k(\eta-2\eta_{\mathrm{r}})| \ll 1$ ，即

$$\mathcal{P}_{\mathrm{T}}(k,\eta) = \frac{2H_k^2\hbar}{\pi^2}8\pi G_{\mathrm{N}}\left[1+2(1-\log 2-\gamma_{\mathrm{E}})\varepsilon\right],$$

$$= \frac{2H_\star^2\hbar}{\pi^2} 8\pi G_{\mathrm{N}} \left[1 + 2\left(1 - \log 2 - \gamma_{\mathrm{E}}\right)\varepsilon - 2\varepsilon \log\left(\frac{k}{k_\star}\right)\right], \tag{117}$$

where γ_{E} is the Euler-Mascheroni constant. Notice that in this limit, the power spectrum does not depend on η , and we made the k -dependence explicit in the second line by expanding H_k around a pivot scale $k_\star$.

其中 γ_{E} 是欧拉-马歇罗尼常数。注意在该极限下，功率谱不依赖于 η ，我们在第二行通过对 H_k 在 pivot 尺度 $k_\star$ 附近展开，显式给出了对 k 的依赖关系。

Current experiments have not been able to detect the primordial gravitational wave background, but the combined (non)-observations of Planck and BICEP experiments allow us to put bounds on the tensor-to-scalar ratio in single-field slow-roll inflation [74]. A discussion of its potential observability in future gravitational wave interferometers and with future CMB experiments can be found in Refs. [73,75,76]. Notice that different models of the very early universe would change the prediction (117): initially excited states [71], temporary departures from the single-field slow-roll scenario [77] or coupling with extra fields [78] might, for instance, be able to generate larger signatures than single-field slow-roll inflation, while modifications of gravity in the high-energy regime could also lead to changes in the spectrum at high frequencies, e.g., through introduction of a cut-off in theories of lower dimensionality in the ultraviolet [79]. Finally, we want to emphasise that the toy model of cosmological evolution of Eq. (102) makes the unrealistic assumption of an instantaneous reheating. Adding a period of reheating is known to significantly modify the resulting spectrum, e.g., the frequency at which it starts to decay, thereby modifying observational perspectives [80].

目前实验尚未探测到原初引力波背景，但普朗克和 BICEP 实验的联合 (未) 观测结果可以给单场慢滚暴胀的张标比施加约束 [74]。关于未来引力波干涉仪和 CMB 实验对原初引力波的可观测性讨论可以参考文献 [73,75,76]。需要注意的是，极早期宇宙的不同模型会改变预言 (117): 例如，初态激发 [71]、对单场慢滚场景的暂时偏离 [77] 或与额外场的耦合 [78]，都可能产生比单场慢滚暴胀更大的可观测信号；而高能区引力的修正也会改变高频段的功率谱，比如在紫外低维理论中引入截断就能产生这种改变 [79]。最后我们需要强调，式 (102) 给出的宇宙演化玩具模型做了瞬时再加热的不现实假设。已知增加一段再加热阶段会显著改变最终谱形，例如改变谱开始衰减的频率，从而改变观测预期 [80]。

Quantum Origin of the Primordial Gravitational Waves

原初引力波的量子起源

To close this part, we want to comment on how quantumness enters the prediction (117).

在本部分的最后，我们要讨论量子性如何进入预言 (117) 中。

First, a subtle point hidden in Eq. (111) is the meaning of the averaging (). In the discussion of the dynamics of perturbations, we have been computing averages in the sense of expectation value for observables in a given quantum state. It is a basic assumption of quantum mechanics that this would be the expected average value of the physical quantity after repeated measurements of it when the system is prepared in the same state. Unfortunately, we only have one realisation of the history of the Universe. Yet, using statistical isotropy, we can treat each (sufficiently large) patch of sky as an independent realisation of the same underlying random process and compute average values over this ensemble of patches. Under an ergodicity assumption, the resulting correlation functions can then be compared to (117), a procedure applied to CMB data analysis [81]. Additional arguments to justify trading quantum averages for classical ones will be discussed in section "Quantum Features in Primordial Gravitational Waves?".

> 首先，式 (111) 中隐藏了一个微妙的问题: 平均 () 的意义是什么。在讨论微扰动力学时，我们计算的平均是指给定量子态下可观测量的期望值。量子力学的基本假设指出，当系统制备在同一状态时，重复测量得到的物理量平均结果就等于这个期望值。遗憾的是，我们只有一份宇宙历史的实现。不过，借助统计各向同性，我们可以将每一块 (足够大的) 天区视为同一基础随机过程的独立实现，对这一天区系综计算平均值。在各态历经假设下，得到的关联函数可以与 (117) 比较，这一方法已经应用于 CMB 数据分析 [81]。我们会在“原初引力波中的量子特性?”一节讨论其他为“量子平均等价于经典平均”辩护的论证。

Second, as we repeatedly emphasised, since the linear evolution is the same in the classical and quantum settings, the quantum aspect has to be confined to the choice of initial state. The result (117) reflects the choice that the waves emerged from initial vacuum fluctuations. For primordial gravitational waves, we are short of observational data to test this prediction. Still, if we were to insist on having a purely classical treatment, then a classical vacuum of gravitational waves, i.e., $a_{\pm\mathbf{k}}(\eta_{\mathrm{in}}) = 0$, would persist throughout the evolution. There would simply be no primordial gravitational waves. On the contrary, initial gravitational waves would be classically amplified by cosmological expansion, but we then have to motivate a specific choice for the initial distribution of perturbations. For scalar perturbations, the CMB observations already demonstrated a tremendously good agreement with the predicted power spectrum $\mathcal{P}_{\mathrm{S}}$ of initial vacuum fluctuations for the modes observed [74, 82]. Giving up on a quantum treatment in the inflationary paradigm would then require providing an ad hoc classical theory that yields the same initial conditions as the quantum vacuum. We could therefore argue that observations of the scalar sector give indirect proof that gravitational degrees of freedom should be quantised.

> 其次，正如我们反复强调的，由于经典框架和量子框架中的线性演化完全相同，量子效应必然仅来源于初态的选择。结果 (117) 反映了我们的选择: 引力波起源于初始真空涨落。对于原初引力波，我们目前缺乏观测数据来检验这个预言。但如果我们坚持采用纯经典处理，那么引力波的经典真空即 $a_{\pm\mathbf{k}}(\eta_{\mathrm{in}}) = 0$ 会在整个演化过程中保持不变，根本不会产生原初引力波。反过来，初始引力波可以被宇宙膨胀经典放大，但我们必须为微扰的初始分布给出一个特定的动机。对于标量微扰，CMB 观测已经证实，观测到的模式的初始真空涨落预言功率谱 $\mathcal{P}_{\mathrm{S}}$ 与观测结果吻合得极好 [74, 82]。如果在暴涨范式中放弃量子处理，就需要构造一个特殊的经典理论，让它给出和量子真空完全相同的初始条件。因此我们可以说，标量 sector 的观测已经间接证明引力自由度应当是量子化的。

Yet, third, it is sometimes argued (see, e.g., Refs. [33, 83]) that the verification of the prediction (117) would provide additional insights on the quantum aspect of gravity with respect to the observation of the scalar perturbations. In the treatment of scalar perturbations in single-field slow-roll inflation, the appropriate gauge-invariant variable is the Mukhanov-Sasaki (MS) field related to the perturbations of the inflaton $\delta\phi$ and the gravitational potential Ψ through

第三，常有观点认为 (例如参见文献 [33, 83])，相较于标量微扰观测，验证预言 (117) 能为引力的量子特性提供更多洞见。在单场慢滚暴涨的标量微扰处理中，合适的规范不变量是穆哈诺夫-佐佐木 (MS) 场，它通过以下式子与暴涨子 $\delta\phi$ 微扰和引力势 Ψ 联系起来

$$v_{\mathrm{MS}} = \frac{z}{\kappa}\left(\Psi + \mathcal{H}\frac{\delta\phi}{\phi_0'}\right), \tag{118}$$

where $z = a\sqrt{2\varepsilon}, \varepsilon$ being the first slow-roll parameter, ϕ_0 the homogeneous background inflaton field and κ the reduced Planck mass. This is a scalar field whose

其中 $z = a\sqrt{2\varepsilon}, \varepsilon$ 是一阶慢滚参数，ϕ_0 是均匀背景暴涨子场，κ 是约化普朗克质量。这是一个标量场，它的

Lagrangian is the same as (22) upon substituting $a \to z$ and up to normalisation. The MS field is then quantised in exactly the same manner as the two polarisations of the graviton and the power spectrum evaluated by initially choosing the Bunch-Davies vacuum. However, in the absence of perturbations $\delta\phi$ for the scalar field, the equation of motion of the scalar part of the metric perturbations shows that they can be set to zero. This is the so-called synchronous gauge [84]. The existence of scalar perturbations then requires the presence of the scalar field $\delta\phi$ and is not intrinsic to the gravitational degrees of freedom. Even when $\delta\phi \neq 0$, in the synchronous gauge $\Psi = 0$, and only the scalar field contributes to the perturbations. In this gauge, the whole quantification process and evaluation of the power spectrum only deal with the physics of a quantum scalar field that is not related to gravitational degrees of freedom. This could therefore cast some doubt on whether some gravitational degrees of freedom were even quantised in the first place. Such ambiguity does not exist when dealing with gravitational waves. In the absence of any anisotropic stress, the gravitational waves persist, and the quantisation procedure is undeniably a perturbative quantisation of the gravitational field. Verification of the prediction (117) would then be an indirect observational proof that the gravitational field must be quantised.

在替换 $a \to z$ 后，拉格朗日量和 (22) 一致，仅归一化不同。MS 场的量子化方式和引力子两个偏振的量子化方式完全相同，并且通过初始选择邦奇-戴维斯真空计算功率谱。但当标量场不存在微扰 $\delta\phi$ 时，度规微扰标量部分的运动方程表明其可以取零，这就是所谓的同步规范 [84]。因此标量微扰的存在依赖于标量场 $\delta\phi$ ，并非引力自由度本身固有的性质。即使在 $\delta\phi \neq 0$ 的情况下，在同步规范 $\Psi = 0$ 中也只有标量场对微扰有贡献。在这个规范下，整个量子化过程和功率谱计算只处理和引力自由度无关的量子标量场物理。因此人们有理由怀疑，引力自由度本身到底有没有被量子化。而这种歧义在研究引力波时并不存在。即使不存在各向异性应力，引力波依然存在，并且量子化过程毫无疑问是对引力场的微扰量子化。因此验证预言 (117) 会成为引力场必须量子化的间接观测证据。

Finally, to mitigate the above discussion, let us also mention an argument against our line of reasoning, as discussed, e.g., in Ref. [35]. The argument made above can be reversed, as one can also find a gauge in which the perturbation of the field vanishes altogether, with $\delta\phi = 0$, while the metric part is $\Psi \neq 0$; in this gauge, the quantisation is then over an element of the metric only. In addition, since perturbations of matter and geometry appear on each side of Einstein equations (however, note that their quantum counterpart is unknown), it is inconsistent to quantise only one degree of freedom. The observational verification of the prediction for the scalar power spectrum thus can be argued to be an indirect proof that the gravitational field should be quantised.

最后，为了平衡上述讨论，我们还要提及一种反对我们推理思路的观点，例如文献 [35] 中就讨论了该观点。上文的论证思路可以反转: 我们也可以找到一个规范，其中场的微扰完全消失，满足 $\delta\phi = 0$ ，而度规部分为 $\Psi \neq 0$ ；在该规范下，仅需对度规分量进行量子化。此外，由于物质微扰和几何微扰分别出现在爱因斯坦方程的两侧 (但需注意，它们的量子对应物尚不明确)，仅对其中一个自由度量子化是不一致的。因此可以说，标量功率谱预测得到观测验证，就是引力场应当被量子化的间接证据。

Quantum Features in Primordial Gravitational Waves?

原初引力波中的量子特征?

Given the quantum origin of primordial gravitational waves, it may seem natural to wonder about their state's quantum or classical character at present. While it is expected that we will never be able to detect the signal produced by a single graviton [85], a discrete spectrum of excitations is not the only specific feature of a quantum theory. For instance, entanglement is a statistical quantum feature that can be experimentally verified using Bell inequalities [86, 87]. The exciting possibility that primordial gravitational waves exhibit such features has been investigated since this idea was first put forward by Grishchuk and Sidorov in Ref. [30]. These discussions have gradually introduced many concepts borrowed from low-energy quantum physics, particularly quantum optics: squeezing, quasi-probability distribution, decoherence, and quantum discord. In this section, we will review this line of research following a historical approach and trying to show the progress brought by each contribution.

鉴于原初引力波起源于量子过程，我们如今自然会好奇其状态究竟是量子还是经典性质。虽然我们大概率永远无法探测到单个引力子产生的信号 [85]，但激发的离散谱并不是量子理论唯一的特有特征。例如，纠缠就是一种可通过贝尔不等式实验验证的统计量子特征 [86, 87]。自格里什丘克和西多罗夫在文献 [30] 中首次提出原初引力波可能存在这类特征后，这一极具吸引力的猜想便得到了研究。这些讨论逐步引入了许多从低能量子物理 (尤其是量子光学) 借来的概念: 压缩、准概率分布、退相干和量子失谐。在本节中，我们将沿着历史脉络回顾这一研究方向，力图展示每项贡献带来的进展。

This section is structured as follows. First, arguments based on the very squeezed character of the state are used to justify a classical treatment to compute cosmological observables [45]. This approach, sometimes called ”decoherence without decoherence” [39], and its critics are reviewed in section ”Classicalisation of Perturbations Without Decoherence”. It turns out, however, that the classicality identified by these works does not do away with all the quantum features of the state; the state of the perturbations could, for instance, violate a Bell inequality [88]. We review these ”quantum information” approaches in section ”Quantum Information Approaches”. Lastly, taking into account the weak interactions of the perturbations is necessary as they would induce decoherence which might erase the quantum features exhibited at the linear level. This aspect is reviewed in section ”Decoherence of Cosmological Perturbations”.

本节结构安排如下。首先，我们介绍基于态的高度压缩性质得到的论证: 该论证支持用经典处理方式计算宇宙学观测量 [45]。我们会在「无退相干的微扰经典化」小节回顾这种有时被称为「无退相干的退相干」[39] 的研究方法及其受到的批评。然而，事实证明这些工作所确认的经典性并没有消除该态的全部量子特征；例如，微扰态仍有可能违反贝尔不等式 [88]。我们会在「量子信息方法」小节回顾这类「量子信息」研究方向。最后，微扰的弱相互作用会诱发退相干，可能抹除线性层面显现的量子特征，因此考虑微扰的弱相互作用是必要的。我们会在「宇宙学微扰的退相干」小节回顾这方面内容。

For the most part, these works are based on analysing the state of a quantum scalar field, which can represent either the MS field of scalar perturbations or one of the polarisations of the tensor perturbations. The mechanisms and arguments being the same for both, we do not distinguish when citing works which refers to which and are only specific when necessary.

这些研究大多基于分析量子标量场的状态，该场既可以代表标量微扰的 MS 场，也可以代表张量微扰的某一种偏振。两种情况的机制和论证都相同，因此在引用相关文献时我们不刻意区分，仅在必要时明确说明。

Classicalisation of Perturbations Without Decoherence

无退相干下微扰的经典化

In Ref. [30], the authors argue that the perturbations exhibit non-classical features due to the fact that the relevant quantum state is strongly squeezed. In order to make the discussion precise yet simple, we focus again on the inflationary period modeled by a de Sitter phase of expansion and assume initial Bunch-Davies vacuum; the relevant equations were derived at the end of section ”Cosmological Evolution”, and the squeezing is shown in Fig. 5. The reasoning can also be extended to certain non-vacuum initial states [89].

在文献 [30] 中，作者认为微扰会表现出非经典特征，因为对应的量子态是高度压缩的。为了让讨论既精确又简洁，我们同样聚焦于由德西特膨胀阶段建模的暴胀期，并假设初始态为邦奇-戴维斯真空；相关方程已在“宇宙演化”小节末尾推导得出，压缩特性可见图 5。该论证也可以推广到某些非真空初始态 [89]。

One of the arguments developed in Ref. [30] is that the trajectory in phase space of a classical system with given initial conditions is represented by a point moving on a single curve. The situation is different for a quantum system. Due to the intrinsic uncertainty stemming from the Heisenberg principle, the trajectory is represented by a moving surface. The quantum state that comes closest to mimicking a classical trajectory would then be a coherent state. Indeed, its trajectory in phase space is represented by a circle moving along a single curve: the system is located within a tube of minimal uncertainty around the classical trajectory. On the contrary, the surface representing an increasingly squeezed state is stretched around its centre delocalising the position of the system away from any single curve. Therefore, they argue a very squeezed state like that of the cosmological perturbations is a very quantum state.

文献 [30] 提出的一个论点是: 给定初始条件的经典系统，其相空间轨迹是一个点沿单条曲线运动。量子系统的情况则不同。由于海森堡原理带来的内禀不确定性，轨迹由一个运动曲面描述。最接近模拟经典轨迹的量子态是相干态。确实，相干态在相空间的轨迹是一个圆沿单条曲线运动: 系统位于经典轨迹周围的最小不确定管内。相反，压缩程度不断增加的量子态对应的曲面包围着中心不断拉伸，使系统位置离域，偏离任何单条曲线。因此他们认为，宇宙学微扰对应的这种高度压缩态是高度量子态。

In a couple of works written in response [39, 45], the authors reproduce and complement the computations made in Ref. [30] but give a different interpretation of the result. The list of their arguments, which we reproduce below, is that the properties of a system in an extremely squeezed state are indistinguishable from that of a classical system whose state is represented by a classic stochastic distribution, an argument borrowed from Ref. [90]. In other words, although the intrinsic quantum uncertainty on the outcome of a measurement dramatically spreads due to the evolution of the system, this uncertainty cannot be distinguished from a purely classical one.

在作为回应的两篇研究 [39, 45] 中，作者重现并补充了文献 [30] 的计算，但对结果给出了不同的诠释。我们下文整理了他们的论点: 处于极端压缩态的系统，其性质无法和服从经典随机分布的经典系统区分开，这一论点源自文献 [90]。换言之，尽管测量结果的内禀量子不确定性会因系统演化大幅扩散，但这种不确定性无法和纯经典不确定性区分开。

To demonstrate this, let us consider the wavefunction of the perturbations in the modes $\pm\mathbf{k}$ decomposed in the R/I sector and given by Eq. (63). Discarding the indices ” k ”, we recall that this is the wavefunction of a one-mode squeezed state. We can show that for large r , it satisfies very well the conditions of the WKB approximation. For a general wavefunction $\Psi(\mu) = C(\mu)\exp[iS(\mu)/\hbar]$, the WKB approximation is valid when the amplitude C varies slowly compared to the phase S : $|\partial S/\partial\mu| \gg |C^{-1}\partial C/\partial\mu|$. Since the WKB approximation is generally understood as a semi-classical limit, this property is sometimes referred to as the ”WKB-classicality” of the state. Using the wavefunction (63), we have

为证明这一点,我们考虑模式 $\pm\mathbf{k}$ 中微扰的波函数,该波函数已按实部/虚部分解,并由式 (63) 给出。忽略下标“k”,我们知道这是单模压缩态的波函数。可以证明,当 r 很大时,它非常好地满足 WKB 近似的条件。对于一般波函数 $\Psi(\mu) = C(\mu)\exp[iS(\mu)/\hbar]$, 当振幅 C 相较于相位 S : $|\partial S/\partial\mu| \gg |C^{-1}\partial C/\partial\mu|$ 变化缓慢时，WKB 近似成立。由于 WKB 近似通常被认为是半经典极限，该性质有时被称为该态的“WKB 经典性”。利用波函数 (63)，我们有

$$C(\mu) = \left(\frac{1}{\hbar\pi\gamma_{11}}\right)^{1/4} e^{-\frac{k\mu^2}{2\hbar\gamma_{11}}}, \tag{119a}$$

$$S(\mu) = k\mu^2\frac{\gamma_{12}}{2\hbar\gamma_{11}}, \tag{119b}$$

where we have dropped the exponent S and the index $\mathbf{k}$ for simplicity. We get

其中为简化起见，我们省略了指数 S 和下标 $\mathbf{k}$ ，可得

$$\left|\frac{C}{\partial C/\partial\mu}\frac{\partial S/\partial\mu}{\hbar}\right| = \left|\sin(2\varphi_k)\sinh(2r_k)\right|. \tag{120}$$

In the de Sitter case, using Eq. (107), one has $\sin(2\varphi_k)\sinh(2r_k) \approx e^N$; the condition is perfectly satisfied. We then compute the action of $\widehat{\mu}$ and $\widehat{\pi}$ on such a state

在德西特情形下，利用式 (107)，可得 $\sin(2\varphi_k)\sinh(2r_k) \approx e^N$ ；该条件完全满足。我们随后计算 $\widehat{\mu}$ 和 $\widehat{\pi}$ 在该态上的作用

$$\widehat{\mu}\Psi(\mu) = \mu\Psi(\mu), \tag{121a}$$

$$\widehat{\pi}\Psi(\mu) = -i\hbar\frac{\partial\Psi}{\partial\mu} = \frac{\partial S}{\partial\mu}\left(1 - i\hbar\frac{\partial C/\partial\mu}{C\partial S/\partial\mu}\right)\Psi(\mu) \approx \frac{\partial S}{\partial\mu}(\mu)\Psi(\mu),$$

(121b)

where in the last line we have used Eq. (120). This last equality suggests that, neglecting sub-dominant contributions, we could attribute an unambiguous value to the ”momentum” π through the relation $\pi \approx \partial S/\partial\mu$ [45] while the value of the position μ would be controlled by the probability distribution given by the μ - representation of the wavefunction, namely,

其中最后一行我们用到了式 (120)。这个等式表明, 忽略次主导贡献后, 我们可以通过关系 $\pi \approx \partial S/\partial\mu$ [45] 给“动量” π 赋予明确值，而位置 μ 的值由波函数的 μ 表示给出的概率分布决定，即

$$P(\mu) = C(\mu)^2 = \left(\frac{k}{\pi\hbar\gamma_{11}}\right)^{1/2} e^{-\frac{k\mu^2}{\hbar\gamma_{11}}}. \tag{122}$$

To make this intuition rigorous, which is not always possible as we explain at the end of the section, we have to use a phase space representation of the state. The Wigner function $W^{\mathrm{S}}(\mu,\pi)$ can be factorised

要让这个直觉变得严谨——正如我们在本节末尾会说明的，这并非总能做到——我们必须使用该态的相空间表示。维格纳函数 $W^{\mathrm{S}}(\mu,\pi)$ 可以因式分解

$$W^{\mathrm{S}}(\mu,\pi) = \sqrt{\frac{k}{\pi\hbar\gamma_{11}}}e^{-\frac{k\mu^2}{\hbar\gamma_{11}}}\sqrt{\frac{\gamma_{11}}{k\pi\hbar}}e^{-\frac{\gamma_{1}1}{\hbar k}\left(\pi-\frac{\gamma_{12}}{\gamma_{11}}k\mu\right)^2}, \tag{123}$$

$$= P(\mu)\sqrt{\frac{\gamma_{11}}{k\pi\hbar}}e^{-\frac{\gamma_{11}}{\hbar k}\left(\pi-\frac{\gamma_{12}}{\gamma_{11}}k\mu\right)^2},$$

where the relation $\det(\gamma) = \gamma_{11}\gamma_{22} - \gamma_{12}^2 = 1$ (we are using a pure state) was used. The first piece is the probability distribution (122). The second piece controls the value of $\pi - \gamma_{12}k\mu/\gamma_{11} = \pi - \partial S/\partial\mu$, i.e., the difference between the actual value of π and that attributed to it following the WKB classicality approach. It can be read out from the above, or shown by a straightforward computation using covariance matrix elements, that

此处已经用到关系 $\det(\gamma) = \gamma_{11}\gamma_{22} - \gamma_{12}^2 = 1$ (我们采用的是纯态)。第一项是概率分布 (122)。第二项控制 $\pi - \gamma_{12}k\mu/\gamma_{11} = \pi - \partial S/\partial\mu$ 的值，即 π 的实际值与 WKB 经典方法得到的值之差。从上式中可以直接读出，也可以通过协变矩阵元进行简单计算证明:

$$\left\langle\left(\widehat{\pi} - \frac{\gamma_{12}}{\gamma_{11}}k\widehat{\mu}\right)^2\right\rangle = \frac{\hbar k}{2}\frac{1}{\gamma_{11}} \approx \frac{\hbar k}{2}e^{-2N}, \tag{124}$$

where we have taken the super-Hubble limit in the last equality. Since the state is Gaussian, and $\hat{\pi}, \hat{\mu}$ are centered, this is the only quantity that controls the error induced by replacing $\hat{\pi}$ by its WKB counterpart $\gamma_{12}k\hat{\mu}/\gamma_{11}$ in the expectation values. As inflation proceeds, this error becomes exponentially small, while the fluctuations of $\hat{\mu}$ get exponentially large, and that of $\gamma_{12}k\hat{\mu}/\gamma_{11}$ tends to a constant. Therefore, to compute the expectation value of any operator which is a polynomial in $\hat{\mu}$ and $\hat{\pi}$, one can safely make the WKB replacement. We emphasise that, to have meaningful operators, the coefficients of these polynomials must not depend on the state of the system. In such polynomials, when expanding $\hat{\pi}$ as $(\hat{\pi} - \gamma_{12}k\hat{\mu}/\gamma_{11}) + \gamma_{12}k\hat{\mu}/\gamma_{11}$, the coefficients of $\hat{\mu}$ and $\hat{\pi}$ cannot conspire to yield an expression depending only on the subdominant combination $\hat{\pi} - \gamma_{12}k\hat{\mu}/\gamma_{11}$ since it explicitly depends on the squeezing parameters. The translation of this approximation in terms of the Wigner function is to take the limit of infinite r_k , with $\gamma_{11} \to \infty$, and to replace the Gaussian over $\pi - \partial S/\partial\mu$ by a Dirac delta [21, 39, 45]

其中我们在最后一个等式中取了超哈勃极限。由于该态是高斯态，且 $\hat{\pi}, \hat{\mu}$ 中心化，这是控制在期望值中将 $\hat{\pi}$ 替换为其 WKB 对应项 $\gamma_{12}k\hat{\mu}/\gamma_{11}$ 所引入误差的唯一物理量。随着暴胀进行，该误差呈指数减小，而 $\hat{\mu}$ 的涨落呈指数增大，$\gamma_{12}k\hat{\mu}/\gamma_{11}$ 的涨落则趋于常数。因此，计算任意关于 $\hat{\mu}$ 和 $\hat{\pi}$ 的多项式算符的期望值时，可以安全地进行 WKB 替换。我们需要强调，要得到有意义的算符，这些多项式的系数不能依赖于系统的态。在这类多项式中，将 $\hat{\pi}$ 展开为 $(\hat{\pi} - \gamma_{12}k\hat{\mu}/\gamma_{11}) + \gamma_{12}k\hat{\mu}/\gamma_{11}$ 后，$\hat{\mu}$ 和 $\hat{\pi}$ 的系数无法共同得到仅依赖于次优组合 $\hat{\pi} - \gamma_{12}k\hat{\mu}/\gamma_{11}$ 的表达式，因为该组合明确依赖于压缩参数。该近似用维格纳函数表述就是: 取无穷大 r_k 极限，满足 $\gamma_{11} \to \infty$ ，并将 $\pi - \partial S/\partial\mu$ 上的高斯分布替换为狄拉克 δ 函数 [21, 39, 45]

$$W^{\mathrm{S}}(\mu, \pi) \approx P(\mu)\,\delta\left(\pi - \frac{\gamma_{12}}{\gamma_{11}}k\mu\right). \tag{125}$$

The interpretation of this equation is straightforward: when computing expectation values using the Wigner function and Eq. (54), up to very sub-dominant contributions, we can replace π by $\partial S/\partial\mu$ in the Weyl transform and take the average on μ using the classical stochastic variable of distribution Eq. (122). In the limit of Eq. (125), the contour levels of the Wigner function are squashed from ellipses to lines, and this implies that the size of the sub-fluctuant mode has been neglected. This line-like limit of the Wigner function is visible in the last panels of Fig. 5.

该方程的解释十分直接: 使用维格纳函数和式 (54) 计算期望值时，在忽略极小次优贡献的前提下，我们可以在魏尔变换中用 $\partial S/\partial\mu$ 替换 π ，再利用分布为式 (122) 的经典随机变量对 μ 取平均。在式 (125) 的极限下，维格纳函数的等高线从椭圆被压缩为直线，这意味着我们忽略了次涨落模式的尺寸。维格纳函数的这种线状极限可以在图 5 的最后几幅面板中看到。

We conclude with a series of remarks on this result. First, it is clear that the replacement $\hat{\pi} \to \partial S/\partial\mu$ cannot be exact as it implies $[\hat{\mu}, \hat{\pi}] = 0 \neq i\hbar$, thus violating the canonical commutation relations, although those must be verified irrespective of the state of the system. Yet, the contribution of this non-vanishing commutator to the expectation value of operators $O(\hat{\mu}, \hat{\pi})$ which are polynomial in $\hat{\mu}$ and $\hat{\pi}$ is negligible.

我们对该结果给出几点总结性评论。首先，显然替换 $\hat{\pi} \to \partial S/\partial\mu$ 不可能是精确的，因为它会导出 $[\hat{\mu}, \hat{\pi}] = 0 \neq i\hbar$ ，从而违反正则对易关系——尽管正则对易关系必须不依赖系统的态成立。但对于任意关于 $\hat{\mu}$ 和 $\hat{\pi}$ 的多项式算符 $O(\hat{\mu}, \hat{\pi})$ ，该非零对易子对算符期望值的贡献可以忽略。

The second remark is that, as explained in Ref. [72], we want to emphasise that the Wigner function of a

WKB state does not in general give rise to a Dirac delta; in fact, it needs not even be positive everywhere. The naive intuition is only verified here because the state is also Gaussian. In addition, the fact that the Wigner function can be negative suggests taking with a grain of salt the idea that any WKB state is understandable as an approximate classical state.

第二点需要说明的是，正如文献 [72] 所解释的，我们要强调:WKB 态的维格纳函数通常不会得到狄拉克 δ 函数；实际上，它甚至不需要处处为正。此处直观成立只是因为该态也是高斯态。此外，维格纳函数可以为负的事实说明，我们不能全盘接受“任何 WKB 态都可理解为近似经典态”的观点。

Thirdly, as stressed in Ref. [83], the distribution (125) has some undesirable features for a Wigner function. For instance, computing the purity using the function (125) and Eq. (54) yields an infinite result. This is obviously incorrect since for any quantum state $p_k \leq 1$, and, in this pure case, we had derive earlier $p_k = 1$. Geometrically, by squeezing the ellipse to a line, one looses the information on the area that encodes the purity and the non-commutation of the variables through the Heisenberg uncertainty principle. This additionally informs us that there exist quantities of interest that crucially depend on the sub-leading contributions that were neglected and so on the sub-fluctuant mode.

第三，正如文献 [83] 所强调的，分布 (125) 作为维格纳函数存在一些不理想的性质。例如，用分布 (125) 和式 (54) 计算纯度会得到无穷大的结果。这显然不正确，因为对任意量子态 $p_k \leq 1$ ，在纯态情况下我们此前已经推导出 $p_k = 1$ 。从几何上看，将椭圆压缩为一条直线会丢失编码纯度的面积信息，也会丢失变量因海森堡不确定性原理不对易的信息。这进一步说明，存在一些关键的感兴趣量，它们依赖于我们此前忽略的次领头阶贡献，因此也依赖于次涨落模式。

The fourth point we want to stress concerns classicality. The part of the argument based on analysing the phase space distribution does not actually require large squeezing to be formulated. Indeed, even before taking any limit, the Wigner function of the state is everywhere positive and obeys the classical equations of motion (59), so that using Eq. (54), any observable can be computed using a classical stochastic distribution.

我们要强调的第四点关乎经典性。基于分析相空间分布的这部分论证实际上并不需要大压缩就可以成立。事实上，即使在取任何极限之前，该态的维格纳函数已经处处为正，且满足经典运动方程 (59)，因此利用式 (54)，任何可观测量都可以通过经典随机分布计算。

As a fifth point, let us note that the above statement has to be made more precise because it hides several subtle points. To start with, as pointed out in Ref. [37], beyond quadratic order, the Weyl transform of an observable $O(\widehat{\mu}, \widehat{\pi})$ is, in general, not obtained by replacing the operators $\widehat{\mu}$ and $\widehat{\pi}$ by the corresponding phase space variables, i.e., $O(\mu, \pi) \neq \widetilde{O}(\mu, \pi)$. For instance

第五点需要注意，上述结论需要更精确的表述，因为它隐藏了几个微妙的问题。首先，正如文献 [37] 指出的，超过二阶之后，可观测量 $O(\widehat{\mu}, \widehat{\pi})$ 的外尔变换通常不能通过将算符 $\widehat{\mu}$ 和 $\widehat{\pi}$ 替换为对应的相空间变量即 $O(\mu, \pi) \neq \widetilde{O}(\mu, \pi)$ 得到。例如

$$\widetilde{\widehat{\mu}_{\mathbf{k}}^2 \widehat{\pi}_{\mathbf{k}}^2} + \widetilde{\widehat{\pi}_{\mathbf{k}}^2 \widehat{\mu}_{\mathbf{k}}^2} = 2\mu_{\mathbf{k}}^2 \pi_{\mathbf{k}}^2 - \hbar \tag{126}$$

so that, using Eq. (54)

因此，利用式 (54)

$$\langle \hat{\mu}_{\mathbf{k}}^2 \hat{\pi}_{\mathbf{k}}^2 + \hat{\pi}_{\mathbf{k}}^2 \hat{\mu}_{\mathbf{k}}^2 \rangle = 2\mathbb{E}\left(\mu_{\mathbf{k}}^2 \pi_{\mathbf{k}}^2\right) - \hbar. \tag{127}$$

This extra $\hbar$ is a contribution of the commutator that the Wigner-Weyl formalism takes into account. Therefore, despite the Wigner function being everywhere positive and acting as a measure in Eq. (54), these terms introduce a slight difference with classical stochastic distributions. The culprit is the Weyl transform of the operators rather than the Wigner function. As argued above, in the large squeezing limit, these extra contributions to the Weyl transform of $\hat{\mu}$ and $\hat{\pi}$ are expected to become negligible. The second subtle point is precisely that these distortions will not become negligible for all observables so that the classicality argument does not apply to these. The fact that certain quantum features persist should not be a surprise since we have shown that the gravitons produced by the evolution remain in entangled pairs in the absence of other interactions [83].

这额外的 $\hbar$ 是对易子的贡献，已经被维格纳-外尔形式体系考虑在内。因此，尽管维格纳函数处处为正，且在式 (54) 中充当测度，这些项仍会带来与经典随机分布的微小差异。问题出在算符的外尔变换，而非维格纳函数本身。正如前文所述，在大压缩极限下，$\hat{\mu}$ 和 $\hat{\pi}$ 的外尔变换的这些额外贡献预计会变得可以忽略。第二个微妙之处在于，并非对所有可观测量这些畸变都会变得可以忽略，因此经典性论证不适用于这些情况。既然我们已经证明，在没有其他相互作用时，演化产生的引力子仍保持纠缠对形式 [83]，那么某些量子特征能够保留也就不足为奇了。

The findings of this section can be summarised as follows: as long as we measure only $\hat{\mu}$ and $\hat{\pi}$, or observables which are polynomials of it, super-Hubble modes behave classically since their expectation values can be completely reproduced by a classical stochastic distribution [37, 88].

本节结论可以总结如下: 只要我们仅测量 $\hat{\mu}$ 和 $\hat{\pi}$，或是它们的多项式可观测量，超哈勃模式就表现出经典行为，因为它们的期望值可以完全由经典随机分布重现 [37, 88]。

Quantum Information Approaches

量子信息方法

It has to be mentioned that the authors of Ref. [45] do recognise the possibility that other operators would exhibit quantum features since squeezed states are known to possess such features in quantum optics experiments. However, they dismiss this possibility by arguing that, contrary to quantum optics, one can only perform measurements of the values of the fields $\hat{\mu}_k$ and $\hat{\pi}_k$ and not, say, of the number of particles $\hat{n}_k$. Therefore the ”decoherence without decoherence” argument is sufficient to claim that the perturbations are practically classical. Setting temporarily aside the question of their observability, we now derive examples of operators revealing non-classicality features in the state of primordial gravitational waves.

必须指出，文献 [45] 的作者确实承认，其他算符可能存在量子特性，因为已知压缩态在量子光学实验中具备这类特性。但他们排除了这种可能性，理由是与量子光学不同，我们只能对场 $\hat{\mu}_k$ 和 $\hat{\pi}_k$ 做测量，无法测量粒子数 $\hat{n}_k$ 这类量。因此，他们认为“无退相干的退相干”论证足以证明原初引力波扰动实际上是经典的。我们暂且搁置可观测性问题，接下来推导能揭示原初引力波态中非经典特性的算符示例。

We have already mentioned that the purity of the state cannot be computed if the sub-dominant contributions of the non-vanishing commutators are dropped. In Ref. [91], the authors showed that in order to correctly compute the entropy of the state using the von Neumann entropy $S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\log(\hat{\rho})]$, the sub-dominant contributions have to be restored. For a two-mode squeezed state, the von Neumann entropy reads [92]

我们已经提到，若舍去非对易子的次主导贡献，就无法计算态的纯度。文献 [91] 的作者证明，若要使用冯·诺依曼熵 $S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\log(\hat{\rho})]$ 正确计算态的熵，必须恢复次主导贡献。对于双模压缩态，冯·诺依曼熵可写为 [92]

$$S(\hat{\rho}) = 2f[\det(\gamma)] \tag{128}$$

where the growing function f is defined for $x \geq 1$ by

其中增长函数 f 对 $x \geq 1$ 定义为

$$f(x) = \left(\frac{x+1}{2}\right)\log_2\left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right)\log_2\left(\frac{x-1}{2}\right). \tag{129}$$

Both the entropy and purity are controlled by the determinant of the covariance matrix, which requires the inclusion of sub-dominant contributions to be correctly evaluated. For the pure two-mode squeezed state of perturbations, one gets $\det(\gamma) = 1$, and the definition of f gives $f(1) = 0$, so we recover that the entropy vanishes.

熵和纯度都由协方差矩阵的行列式决定，要正确计算行列式必须包含次主导贡献。对于纯扰动双模压缩态，可得 $\det(\gamma) = 1$ ，结合 f 的定义可得 $f(1) = 0$ ，因此我们得到熵为零的结果。

We have so far only shown that, for certain operators, it is not appropriate to neglect the sub-fluctuant mode. We now go further and exhibit quantities whose values cannot be accounted for if the system is described by a classical stochastic distribution. The prime example of such quantities is the combinations of expectation values of spin operators entering the famous Bell inequalities [86]. To design a Bell inequality, one has to exhibit a combination of operators $C\left(\hat{O}_1, \dots, \hat{O}_n\right)$ such that, if the expectation values of the $\hat{O}_i$ s are described by a stochastic probability distribution, then C is bounded by a real number c

到目前为止我们仅证明，对于某些算符，忽略次涨落模是不恰当的。我们现在进一步说明，如果用经典随机分布描述该系统，就无法解释一些物理量的值。这类量的典型例子就是著名贝尔不等式中自旋算符期望值的组合 [86]。要构造贝尔不等式，需要得到算符组合 $C\left(\hat{O}_1, \dots, \hat{O}_n\right)$ ，若 $\hat{O}_i$ s 的期望值可以用随机概率分布描述，则 C 被实数 c 界定

$$C\left(O_1,\ldots,O_n\right) \le c. \tag{130}$$

More precisely, the assumption is that their values are described by a local realistic theory. For a discussion of this subtle and important point, we refer to Ref. [93]. As a consequence, if a quantum state is such that $\left\langle C\left(\hat{O}_1,\ldots,\hat{O}_n\right)\right\rangle > c$, then we have proven that not all expectation values of this state can be accounted for by a classical probabilistic theory.

更准确地说，这个假设基于量的值可以定域实在论描述。关于这个微妙且重要的问题，相关讨论参见文献 [93]。因此，若一个量子态满足 $\left\langle C\left(\hat{O}_1,\ldots,\hat{O}_n\right)\right\rangle > c$ ，就证明该态并非所有期望值都可以用经典概率理论解释。

A necessary condition for a state to violate a Bell inequality is that it is not separable [94]. A state $\hat{\rho}$ of a system that can be partitioned in two subsystems A and B is said to be separable in this partition if its density matrix can be written as

一个态满足贝尔不等式破缺的必要条件是它不可分离 [94]。一个系统的态 $\hat{\rho}$ 可以划分为两个子系统 A 和 B ，如果它的密度矩阵可以写成如下形式，就称该态在这个划分下是可分离的

$$\hat{\rho} = \sum_i p_i \hat{\rho}_{\mathrm{A}}^i \bigotimes \hat{\rho}_{\mathrm{B}}^i, \tag{131}$$

where $p_i \ge 0$ and $\sum_i p_i \ge 0$. Such a state can be constructed using a classical protocol [94]. The interpretation of Eq. (131) is that p_i is the probability of finding the system in the sector $\hat{\rho}_{\mathrm{A}}^i \bigotimes \hat{\rho}_{\mathrm{B}}^i$ where the subsystems A and B are independent since the density matrix is factorised. The correlations between the subsystems are thus controlled only by the probabilities $\{p_i\}$ and deemed classical. Nonseparable states are generally called entangled states. In general, it is very difficult to determine whether a state is separable. Fortunately, for Gaussian states, the Peres-Horodecki criterion allows us to check separability using the covariance matrix elements only [95]. This method was first applied to cosmological perturbations by Campo and Parentani in Ref. [44]. We explain their result in the terms used in this review.

其中 $p_i \ge 0$ 和 $\sum_i p_i \ge 0$ 。该态可通过经典协议构造 [94]。对式 (131) 的解释是， p_i 是系统处于区 $\hat{\rho}_{\mathrm{A}}^i \bigotimes \hat{\rho}_{\mathrm{B}}^i$ 的概率，在该区中子系统 A 与 B 相互独立，因为密度矩阵是可因式分解的。因此子系统间的关联仅由概率 $\{p_i\}$ 控制，属于经典关联。不可分离的态通常被称为纠缠态。一般而言，判断一个态是否可分离非常困难。幸运的是，对于高斯态，佩雷斯-霍罗德基判据允许我们仅使用协方差矩阵元即可检验可分性 [95]。该方法最早由坎波和帕兰塔尼应用于宇宙学微扰研究，见文献 [44]。我们采用本文综述使用的术语解释他们的结论。

We first need to choose a partition of the system. The separable character of the state depends on the subsystems considered; for a general discussion of the notion of partition, see Ref. [46]. Using the vectors of conjugate operators introduced in section ”Wigner Function”, we define a (bi)partition of the system by sorting the operators into two vectors of smaller dimensions

我们首先需要对系统进行划分。态的可分性取决于所选取的子系统；关于划分概念的一般性讨论参见文献 [46]。利用“维格纳函数”一节引入的共轭算符矢量，我们通过将算符整理为两个更低维度的矢量，定义了系统的二分划分

$$\hat{X} = \hat{X}_A \bigoplus \hat{X}_B. \tag{132}$$

To represent the state of the perturbations, we have used the R/I partition defined by $\hat{X}_{\mathrm{R/I}} = \left(k^{1/2}\hat{\mu}_k^{\mathrm{R}}, k^{-1/2}\hat{\pi}_k^{\mathrm{R}}, k^{1/2}\hat{\mu}_k^{\mathrm{I}}, k^{-1/2}\hat{\pi}$ where the two subsystems decouple. These operators will, however, mix the creation/annihilation operators (28) defining the modes $\pm\mathbf{k}$. If we are interested in the correlations between these modes, we have to build separate Hermitian operators describing the mode $\mathbf{k}$ and $-\mathbf{k}$. This is readily done by considering

为了表示微扰的态，我们使用了由 $\hat{X}_{\mathrm{R/I}} = \left(k^{1/2}\hat{\mu}_k^{\mathrm{R}}, k^{-1/2}\hat{\pi}_k^{\mathrm{R}}, k^{1/2}\hat{\mu}_k^{\mathrm{I}}, k^{-1/2}\hat{\pi}_k^{\mathrm{I}}\right)$ 定义的实部/虚部 (R/I) 划分，两个子系统在该划分下退耦。但这些算符会混合定义模式 $\pm\mathbf{k}$ 的产生/湮灭算符 (28)。如果我们要研究这些模式之间的关联，就需要构造描述模式 $\mathbf{k}$ 和 $-\mathbf{k}$ 的独立厄米算符。这可以很方便地通过考虑下式得到

$$\hat{q}_{\pm\mathbf{k}} = \sqrt{\frac{\hbar}{2k}}\left(\hat{a}_{\pm\mathbf{k}} + \hat{a}^{\dagger}_{\pm\mathbf{k}}\right) \text{ and } \hat{p}_{\pm\mathbf{k}} = -i\sqrt{\frac{\hbar k}{2}}\left(\hat{a}_{\pm\mathbf{k}} + \hat{a}^{\dagger}_{\pm\mathbf{k}}\right). \tag{133}$$

These operators define the $\pm\mathbf{k}$ partition $\hat{X}_{\pm\mathbf{k}} = \left(k^{1/2}\hat{q}_{\mathbf{k}}, k^{-1/2}\hat{p}_{\mathbf{k}}, k^{1/2}\hat{q}_{-\mathbf{k}}, k^{-1/2} \quad \hat{p}_{-\mathbf{k}}\right)$. We compute the covariance matrix in this partition

这些算符定义了 $\pm\mathbf{k}$ 划分 $\hat{X}_{\pm\mathbf{k}} = \left(k^{1/2}\hat{q}_{\mathbf{k}}, k^{-1/2}\hat{p}_{\mathbf{k}}, k^{1/2}\hat{q}_{-\mathbf{k}}, k^{-1/2} \quad \hat{p}_{-\mathbf{k}}\right)$。我们计算该划分下的协方差矩阵

$$\gamma = \begin{pmatrix} \gamma_{\mathbf{k}} & \gamma_{\mathbf{k},-\mathbf{k}} \\ \gamma_{-\mathbf{k},\mathbf{k}} & \gamma_{-\mathbf{k}} \end{pmatrix}, \tag{134}$$

with

其中

$$\gamma_{\mathbf{k}} = \gamma_{-\mathbf{k}} = \cosh\left(2r_k\right)\mathbb{I}_2 = \left(n_k + \frac{1}{2}\right)\mathbb{I}_2, \tag{135}$$

where $\mathbb{I}_2$ is the two-dimensional identity matrix and

其中 $\mathbb{I}_2$ 是二维单位矩阵，且

$$\gamma_{\mathbf{k},-\mathbf{k}} = \gamma_{-\mathbf{k},\mathbf{k}} = -\sinh\left(2r_k\right)\begin{pmatrix} \cos 2\varphi_k & \sin 2\varphi_k \\ \sin 2\varphi_k & -\cos 2\varphi_k \end{pmatrix} = \begin{pmatrix} \Re\mathrm{e}\left(c_k\right) & \Im\mathrm{m}\left(c_k\right) \\ \Im\mathrm{m}\left(c_k\right) & -\Re\mathrm{e}\left(c_k\right) \end{pmatrix}.$$

(136)

Unlike in the R/I partition, this covariance matrix is not block diagonal. It shows that the $\mathbf{k}$ and $-\mathbf{k}$ particles are correlated. The Peres-Horodecki applied to this covariance matrix reduces to [44]

与实部/虚部 (R/I) 划分不同，该协方差矩阵不是分块对角矩阵。这说明 $\mathbf{k}$ 和 $-\mathbf{k}$ 粒子存在关联。将佩雷斯-霍罗德基判据应用于该协方差矩阵后可化简为 [44]

$$\hat{\rho} \text{ separable in } \pm\mathbf{k} \text{ partition } \Leftrightarrow |c_k| \leq n_k\,. \tag{137}$$

This criterion lends itself to a very simple interpretation; the state will be separable if and only if the correlation of the pairs is larger than their number. When is this satisfied? The condition (137) is straightforwardly expressed in terms of the squeezing parameters. We find that the state is separable if only if $e^{-r_k} \geq 1$, i.e., for the vacuum $r_k = 0$. Therefore, the primordial graviton pairs $\pm\mathbf{k}$ are always entangled. We have found a first quantum feature of their distribution. Notice that the same analysis could be repeated in the R/I partition, but since these sectors are not correlated, it would trivially lead to the conclusion that the state is always separable in this partition. This illustrates clearly the dependence of the (non)- separable character of the state on the choice of subsystems.

该判据的解释非常简单；当且仅当粒子对的关联大于其数量时，态是可分的。那这个条件何时满足？条件 (137) 可以很直接地用压缩参数表示。我们发现，当且仅当 $e^{-r_k} \geq 1$ 时态可分，即对于真空 $r_k = 0$ 满足该条件。因此，原初引力子对 $\pm\mathbf{k}$ 始终是纠缠的。我们由此发现了其分布的第一个量子特征。注意，相同的分析也可以在实部/虚部 (R/I) 划分中重复，但由于这些区间不存在关联，该划分会平凡地得出“态始终可分”的结论。这清晰地说明，态的 (非) 可分性依赖于子系统的选取。

The state of the perturbations we have considered so far is pure. It was shown that, for any entangled pure state, one can build a Bell inequality that the state violates [94]. The separability criterion is, in this case, sufficient. How can we find operators able to violate a Bell inequality for the gravitons? The considerations of section ”Classicalisation of Perturbations Without Decoherence” already demonstrated that, in order to reveal the quantumness of the distribution, we have to use operators which are non-polynomials in $\hat{\mu}_k^{\mathrm{S}}$ and $\hat{\pi}_k^{\mathrm{S}}$. In Ref. [96], Revzen further introduces a distinction between what he calls proper and improper operators.

我们此前讨论的扰动态是纯态。已有研究证明，对于任意纠缠纯态，都可以构造出一个被该态违反的贝尔不等式 [94]。在这种情况下，可分性判据是充分的。我们要如何找到能够让引力子的贝尔不等式产生违反的算符呢? “无退相干扰动经典化”一节的讨论已经表明，要揭示分布的量子特性，我们必须使用关于 $\hat{\mu}_k^{\mathrm{S}}$ 和 $\hat{\pi}_k^{\mathrm{S}}$ 的非多项式算符。在文献 [96] 中，Revzen 进一步区分了他所说的真算符和非真算符。

Proper operators are defined as those that cannot be used to violate a CSH-type [97] Bell inequality when the Wigner function of the state is positive. He shows that any operator $\hat{O}$ whose Weyl transform $\widetilde{O}$ takes values in the set of its eigenvalues is proper. Indeed, the Wigner function then provides an appropriate local hidden variable theory to describe its expectation values. Therefore, we have to use operators that do not fall in this category to build a Bell inequality that can be violated by primordial gravitational waves. In fact, these operators are not uncommon. Consider, for example, the number operator

真算符的定义是: 当态的维格纳函数为正时，无法用它违反 CSH 型 [97] 贝尔不等式。他证明，任意满足其外尔变换 $\widetilde{O}$ 取值落在自身本征值集合内的算符 $\hat{O}$ 都是真算符。此时维格纳函数确实可以给出合适的定域隐变量理论来描述其期望值。因此，我们必须使用不属于此类的算符，才能构造出可被原初引力波违反的贝尔不等式。实际上这类算符并不少见，例如粒子数算符

$$\hat{n}_k = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} = \frac{k}{2\hbar}\widehat{\mu_{\mathbf{k}}}^2 + \frac{1}{2\hbar k}\widehat{\pi_{\mathbf{k}}}^2 + \frac{1}{2}. \tag{138}$$

It has a discrete spectrum, while its Weyl transform $\widetilde{n_k} = \frac{k}{2\hbar}\mu_{\mathbf{k}}^2 + \frac{1}{2\hbar k}\pi_{\mathbf{k}}^2 + \frac{1}{2}$ is a continuous function of the phase space variables. In Ref. [44], Campo and Parentani were the first to exhibit Bell inequalities violated by cosmological perturbations. They emphasised the necessity to use non-polynomial operators in the field operator, and they used as a building block the probability of finding the system in a certain two-mode coherent state

它具有离散谱，但其外尔变换 $\widetilde{n_k} = \frac{k}{2\hbar}\mu_{\mathbf{k}}^2 + \frac{1}{2\hbar k}\pi_{\mathbf{k}}^2 + \frac{1}{2}$ 是相空间变量的连续函数。在文献 [44] 中，Campo 和 Parentani 首次展示了被宇宙学扰动违反的贝尔不等式。他们强调必须使用场算符中的非多项式算符，并将系统处于某特定双模相干态的概率作为构造基础

$$Q(v,w) = \mathrm{Tr}\left(\hat{\rho}\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}\right)$$

$$= \frac{1}{\Delta_k}\exp\left\{-\frac{1}{\Delta_k}\left[(n_k+1)\left(|v|^2+|w|^2\right) - 2\Re\mathrm{e}\left(c_k^* v w\right)\right]\right\}, \tag{139}$$

where $\Delta_k = (n_k+1)^2 - |c_k|^2$, and $\widehat{\Pi}_{k,-k} = |v,\mathbf{k}\rangle\langle v,\mathbf{k}| \otimes |w,-\mathbf{k}\rangle\langle w,-\mathbf{k}|$ projects the subsystem $\mathbf{k}$ (respectively, $-\mathbf{k}$) on the coherent state associated to $v \in \mathbb{C}$ (resp., $w \in \mathbb{C}$) . The bounds given on n_k and c_k in section "Squeezed States" ensure that Δ_k is a positive quantity. This real and positive function of v and w is called the Husimi Q-representation of the state [98]. Like the Wigner function, it is a phase space representation of the state but using coherent states as a basis rather than eigenstates of the field operators. The authors discuss the quantumness of the perturbation using its properties and that of the related Glauber Sudarshan P-representation. They argue that the state not admitting a P-representation can be considered a nonclassical feature. For brievity, we will not discuss these aspects here and refer to Refs. [44, 98] for details. For the purpose of building a Bell inequality, it can be simplified by re-parametrising the arbitrary phase of v to absorb that of c_k . We take $\arg v = 2\arg c_k$ so that $2\Re\mathrm{e}\left(c_k^* v w\right) = 2|c_k|\Re\mathrm{e}\left(v^* w\right)$. For a two-mode squeezed state, $|c_k| = \sqrt{n_k(n_k+1)}$ so that, upon rearranging,

其中 $\Delta_k = (n_k+1)^2 - |c_k|^2$ ，且 $\widehat{\Pi}_{k,-k} = |v,\mathbf{k}\rangle\langle v,\mathbf{k}| \otimes |w,-\mathbf{k}\rangle\langle w,-\mathbf{k}|$ 将子系统 $\mathbf{k}$ (对应 $-\mathbf{k}$) 投影到与 $v \in \mathbb{C}$ (对应 $w \in \mathbb{C}$)) 关联的相干态上。“压缩态”一节给出的 n_k 和 c_k 的范围保证了 Δ_k 是正量。这个关于 v 和 w 的实值正函数被称为该态的胡西米 Q 表象 [98]。和维格纳函数一样，它是态的相空间表象，但使用相干态而非场算符的本征态作为基。作者利用该表象以及相关的格劳贝尔-苏达善 P 表象的性质讨论了扰动的量子特性。他们认为，不支持 P 表象的态可被视为具有非经典特性。为简洁起见，我们在此不讨论这些方面，细节可参见文献 [44, 98]。为了构造贝尔不等式，可以通过重新参数化 v 的任意相位来吸收 c_k 的相位，从而简化问题。我们取 $\arg v = 2\arg c_k$ ，因此得到 $2\Re\mathrm{e}\left(c_k^* v w\right) = 2|c_k|\Re\mathrm{e}\left(v^* w\right)$ 。对于双模压缩态，$|c_k| = \sqrt{n_k(n_k+1)}$ ，重新整理后可得

$$Q(v,w) = \frac{1}{n_k+1}\exp\left(-\frac{|v|^2}{n_k+1}\right)\exp\left(-\left|w - v\sqrt{\frac{n_k}{n_k+1}}\right|^2\right). \tag{140}$$

Since the Husimi representation is also the expectation value of an operator, it can be used in a Bell inequality. The authors then use the Bell inequality demonstrated by Ref. [99] over $Q(v,w)$

由于 Husimi 表示也是算符的期望值，因此可用于贝尔不等式中。随后作者将文献 [99] 证明的贝尔不等式应用于 $Q(v,w)$

$$C(v,w) = \left[Q(0,0) + Q(v,0) + Q(0,w) - Q(v,w)\right]\left(\frac{n_k+1}{2}\right) \leq 1.$$

(141)

They argue that C is maximal for $w = -v$ in which case it only depends on $|v|^2$ and

他们指出，对于 $w = -v$ ，C 取得最大值，此时 C 仅依赖于 $|v|^2$ 且

$$C_{\max}\left(|v|^2\right) = \frac{1}{2}\left[1 + 2e^{-|v|^2} - e^{-2\left(1+\sqrt{\frac{n_k}{n_k+1}}\right)|v|^2}\right]. \tag{142}$$

One can show that, provided we are not in the vacuum $n_k = 0, C_{\max}$ is always larger than unity in the vicinity of $v = 0$, as illustrated in Fig. 7; the Bell inequality is violated. As expected, we have recovered the separability condition. In a later work [100], the authors proved that another inequality, built using operators, also defined in Ref. [101], that are complementary (in the sense that their sum is the identity) to the projectors $\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}$, is violated. They also built other inequalities using the (GKM and Larsson) pseudo-spin operators in the same work. They explicitly showed that all these operators belong to the subclass of improper operators identified by Revzen. Since the Weyl transform of the identity is just the number 1, we can infer from their complementary with the projectors $\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}$ that the operators $\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}$ also belong to this subclass.

可以证明，若我们不处于真空态，在 $v = 0$ 附近 $n_k = 0, C_{\max}$ 始终大于 1，如图 7 所示；因此贝尔不等式被违反。如预期一样，我们得到了可分离性条件。在后续工作 [100] 中，作者证明，另一个利用文献 [101] 中定义的算符构造的不等式也被违反，这些算符与投影算符 $\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}$ 互补 (即二者之和为单位算符)。在同一工作中，作者还利用 (GKM 和 Larsson) 赝自旋算符构造了其他不等式。他们明确证明所有这些算符都属于 Revzen 识别出的非真算符子类。由于单位算符的外尔变换就是常数 1，结合它们与投影算符 $\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}$ 的互补性，我们可以推得算符 $\widehat{\Pi}_{\mathbf{k},-\mathbf{k}}$ 也属于该子类。

We now introduce a last non-classicality criterion, the quantum discord. We start by giving the intuition behind its definition and reviewing some important properties. Technical details in definitions and proofs are skipped and can be found in Refs. [102, 103]. The idea of quantum discord is also to show that correlations between two subsystems are stronger than allowed classically. Two measures of the information attached to these correlations are introduced to that end. These measures are based on the von Neumann entropy, which, as we have shown, is highly sensitive to terms that can be neglected when computing field expectation values. The first measure is the mutual information

现在我们介绍最后一个非经典性判据: 量子失协。我们首先阐释其定义背后的直观物理，并回顾一些重要性质。定义和证明中的技术细节在此略去，可参考文献 [102, 103]。量子失协的核心思想同样是证明两个子系统之间的关联比经典允许的更强。为此，我们引入了两种度量这些关联所含信息的方式，它们都基于冯·诺依曼熵——正如我们已经说明的，冯·诺依曼熵对计算场期望值时可忽略的项高度敏感。第一种度量是互信息

$$\mathcal{I}(A,B) = S(A) + S(B) - S(A,B), \tag{143}$$

where $S(A,B)$ is the von Neumann entropy of the full system while $S(A)$ and $S(B)$ are the entropies of the subsystems. The latter are defined by computing the entropy of the reduced density matrices when one of the subsystems is traced out, e.g., $\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho})$ for the subsystem A . They are also called the entanglement entropy of the state. The second measure is

其中 $S(A,B)$ 是整个系统的冯·诺依曼熵，$S(A)$ 和 $S(B)$ 是两个子系统的熵。子系统熵通过对其中一个子系统求迹得到约化密度矩阵，再计算其熵得到，例如子系统 A 对应的熵为 $\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho})$。这类熵也被称为该量子态的纠缠熵。第二种度量是

$$\mathcal{J}(A,B) = S(A) - S(A \mid B), \tag{144}$$

where $S(A \mid B)$ measures the information gained on A by measuring B. Its precise definition in the quantum setting must therefore include the system state after measuring the system B. It is obtained by minimising the density matrix residual entropy after having measured a complete set of projections on B, i.e., by maximising the information gain. For a quantum state, we then define the quantum discord as their difference

其中 $S(A \mid B)$ 度量测量 B 后得到的关于 A 的信息量。因此在量子框架下，它的精确定义必须包含测量 B 后的系统态。它通过对 B 测量一组完备投影后，最小化密度矩阵的剩余熵得到，也就是通过最大化信息增益得到。对于一个量子态，我们将量子失协定义为二者之差

$$\mathcal{D}(A,B) = \mathcal{I}(A,B) - \mathcal{J}(A,B), \tag{145}$$

which is shown to be in general non-negative. The key observation is that, by the Bayes theorem, $\mathcal{I}$ and $\mathcal{J}$ coincide for a classical system so that the discord vanishes. A non-vanishing discord $\mathcal{D}(A,B) > 0$ is therefore taken as a nonclassical feature. As the other criteria introduced, the quantum discord depends on the choice of partition $\widehat{R} = \widehat{R}_A \bigoplus \widehat{R}_B$. However, it does not depend on the operators chosen to represent them, i.e., it is invariant under any change of operators within the sectors A and B. We call such a quantity a local symplectic invariant. On the contrary, a Bell inequality is not necessarily a local symplectic invariant. A last important property of the discord is that, for a pure state, it reduces to the entanglement entropy $\mathcal{D}(A,B) = S(A) = S(B)$, and, for a pure state still, being entangled is equivalent to a non-vanishing entanglement entropy. Therefore, all criteria introduced (separability, Bell inequality, quantum discord) are equivalent for pure states. The cosmological perturbations must therefore have a non-vanishing quantum discord.

已证明该式总体为非负。关键结论是: 根据贝叶斯定理，对于经典系统，$\mathcal{I}$ 与 $\mathcal{J}$ 一致，因此量子失协为零。因此，非零的量子失协 $\mathcal{D}(A,B) > 0$ 被认定为非经典特征。和其他引入的判据一样，量子失协依赖于划分 $\widehat{R} = \widehat{R}_A \bigoplus \widehat{R}_B$ 的选择。但它不依赖于用于表示它们的算符，即在分区间 A 和 B 内更换任意算符，它都保持不变。我们将这类量称为局部辛不变量。相反，贝尔不等式不一定是局部辛不变量。量子失协最后一个重要性质是: 对于纯态，它退化为纠缠熵 $\mathcal{D}(A,B) = S(A) = S(B)$，且仍对纯态成立，存在纠缠等价于纠缠熵非零。因此，所有引入的判据 (可分性、贝尔不等式、量子失协) 对纯态而言是等价的。故宇宙学扰动必然具有非零量子失协。

The quantum discord of cosmological perturbations was computed in Ref. [37] for the $\pm\mathbf{k}$ partition. It was already used in a work on cosmological perturbations [104], but the author considered correlations of another nature, namely, that of the perturbations and their environment. It reads

文献 [37] 针对 $\pm\mathbf{k}$ 划分计算了宇宙学扰动的量子失协。它早已被用于一篇关于宇宙学扰动的工作 [104]，但作者讨论的是另一种性质的关联，即扰动与其环境之间的关联。其表达式为

$$\mathcal{D}_{\pm\mathbf{k}} = f\left[\cosh\left(2r_k\right)\right], \tag{146}$$

where f was defined in Eq. (129). We immediately verify that the discord is nonvanishing provided that $r_k > 0$, i.e., that we are not in the vacuum. Taking the de Sitter limit of the above expression, we find $\mathcal{D}_{\pm k} \approx 2r_k/\ln 2 \approx 2N/\ln 2$; the discord grows linearly with the number of e -folds.

其中 f 已在式 (129) 中定义。我们可直接验证，只要 $r_k > 0$ (即我们不处于真空态)，量子失协就不为零。对上述表达式取德西特极限，我们得到 $\mathcal{D}_{\pm k} \approx 2r_k/\ln 2 \approx 2N/\ln 2$ ；量子失协随 e 暴胀倍数线性增长。

The results of this section demonstrate that, as suspected, the primordial gravitational waves are only classical if we restrict our attention to field operators $\hat{\mu}$ and $\hat{\pi}$. We showed, using several criteria, that their state exhibits in principle quantum features: it is entangled, violates Bell inequalities and has a non-vanishing quantum discord. We additionally verified that these three criteria are equivalent for pure states like the two-mode squeezed state considered here. Still, in any realistic model of the early Universe, this assumption of purity has to be given up. What has allowed us so far to simply consider a couple of modes $\pm\mathbf{k}$ of the field is that we have neglected all interactions of the gravitational waves, in particular their intrinsic nonlinearities. We were justified in doing since the latter are weak. Yet, it is well known that even very weak interactions can lead to an erasure of non-classical features by inducing decoherence of the system. The most famous example of this is probably that a grain of dust whose spatial superposition would be turned into a classical superposition in a fraction of an instant simply by the scattering of photons from the CMB [105]. The importance of decoherence in the discussion of quantum features of cosmological perturbations was quickly realised [106, 107]. We now investigate how it affects the state, in general, and in particular the quantum features we have just exhibited.

本节结果表明，正如猜想的那样，只有当我们将研究范围限定在场算符 $\hat{\mu}$ 和 $\hat{\pi}$ 时，原初引力波才是经典的。我们通过多个判据证明，其状态原则上存在量子特征: 它是纠缠的，违反贝尔不等式，且具有非零量子失协。我们还验证了，对于本文讨论的双模压缩态这类纯态，这三个判据是等价的。但在任何 realistic 的早期宇宙模型中，都必须放弃纯态假设。迄今为止我们之所以能仅考虑场的一对模式 $\pm\mathbf{k}$ ，是因为我们忽略了引力波的所有相互作用，尤其是其内在非线性。由于这类相互作用很弱，我们的处理是合理的。但众所周知，即使极弱的相互作用也会诱发系统退相干，抹除非经典特征。最著名的例子大概是: 尘埃颗粒的空间叠加会因 CMB 光子的散射，在瞬间转变为经典叠加 [105]。人们很快意识到退相干在讨论宇宙学扰动量子特征中的重要性 [106, 107]。下面我们研究退相干如何影响系统整体，尤其是我们刚刚揭示的量子特征。

Decoherence of Cosmological Perturbations

宇宙学微扰的退相干

We start by briefly recalling some basic concepts of decoherence and refer to Ref. [108] for details. The two-mode squeezed state of a couple of modes $\pm\mathbf{k}$ is a pure state represented by the ket (46). One can easily compute its density matrix and express it in the graviton two-mode number basis

我们首先简要回顾退相干的一些基本概念，细节参见文献 [108]。一对模 $\pm\mathbf{k}$ 的双模压缩态是可以由右矢 (46) 表示的纯态。我们可以很容易地计算它的密度矩阵，并在引力子双模粒子数基下将其表示为

$$\hat{\rho}_{2\mathrm{MSS}}=\frac{1}{\cosh^{2}(2r_{k})}\sum_{n,n'=0}^{+\infty}[-\tanh(2r_{k})]^{n+n'}e^{2i(n-n')\varphi_{k}}\left|n_{\mathbf{k}},n_{-\mathbf{k}}\right\rangle\left\langle n'_{\mathbf{k}},n'_{-\mathbf{k}}\right|. \tag{147}$$

The coefficients on the diagonal $q_n = \tanh^{2n}(2r_k)/\cosh^2(2r_k)$ give a classical probability distribution over the two-mode number states, while the non-diagonal reflects the quantum interferences between them. If we discard these terms, the density matrix reads

对角元系数 $q_n = \tanh^{2n}(2r_k)/\cosh^2(2r_k)$ 给出了双模粒子数态上的经典概率分布，非对角元则反映了它们之间的量子干涉。如果我们去掉这些非对角项，密度矩阵可写为

$$\hat{\rho}_{\mathrm{th.}}=\frac{1}{\cosh^{2}(2r_{k})}\sum_{n=0}^{+\infty}\tanh^{2n}(2r_{k})\left|n_{\mathbf{k}},n_{-\mathbf{k}}\right\rangle\left\langle n_{\mathbf{k}},n_{-\mathbf{k}}\right|. \tag{148}$$

The state now represents a classical superposition of different number states with the same probabilities as $\hat{\rho}_{2\mathrm{MSS}}$. Such states are called statistical mixtures and are indeed mixed states (except if all coefficients but one vanish) since $p_k = \sum_n q_n^2$ and $q_n \leq 1$. The general idea of decoherence is that interactions of the system with a large number of unobserved degrees of freedom, referred to as the environment, precisely diagonalise the density matrix, driving the state to a statistical mixture. Equation (148) is actually the density matrix of a thermal state with, on average, n_k particles in both modes. Since it is fully diagonal, it is considered the result of a complete decoherence process. A very important point is that the (non)-diagonal character of the matrix depends on the basis, e.g., the matrix is originally diagonal in the two-mode squeezed state basis. The basis in which decoherence makes the density matrix diagonal is called the pointer basis. Once again, we see that the choice of basis and operators to analyse the state of the system is crucial. For cosmological perturbations, several pointer basis were considered: coherent state basis [44, 109], field amplitude basis [106, 107, 110, 111], number basis [110] and others [112]. Notice that some of these works [110, 111] predate those referred to in the last section. Decoherence was, in fact, already investigated in the context of the early Universe before the argument of "decoherence without decoherence" was made. It was especially used to try to make sense of the solutions of quantum cosmology, where both the background and the perturbations are treated as quantum fields [113]. Ultimately, in a realistic model, the pointer basis is given by the eigenstates of the interaction Hamiltonian selected. The basis thus bears a double physical sense: it tells us for which type of measurements the system appears classical, e.g., measures of field amplitude or of number of particles, and also to which operators of the system is the environment sensitive. In their follow-up articles Refs. [106, 107] to Ref. [39], the group of authors (Kiefer, Lesgourgues, Starobinski, Polarski) considered the effect of decoherence. They argued that the correct pointer basis should be the field amplitude basis on the ground that self-interactions of pure gravity are local in the field basis, i.e., $H_{\mathrm{in}} \propto \hat{\mu}^n(\mathbf{x},\eta)\,\hat{\pi}^m(\mathbf{x},\eta)$. Since these interactions are contained in the Einstein-Hilbert action, they constitute a minimal and well-defined source of decoherence. They were then taken into account in a more realistic model of decoherence for the first time in Refs. [114, 115]. There, the system considered is made up of the observed large wavelengths, while the environment is made up of the rest of the short, unobserved, wavelengths like in stochastic inflation [116]. This approach was originally performed for scalar perturbations and was later generalised to tensor perturbations [117].

此时该态代表不同粒子数态的经典叠加，概率与 $\hat{\rho}_{\mathrm{2MSS}}$ 相同。这类态被称为统计混合态，确实是混合态 (除非只有一个系数非零)，因为 $p_k = \sum_n q_n^2$ 和 $q_n \leq 1$ 。退相干的核心思想是: 系统与大量未被观测的自由度 (即环境) 的相互作用会恰好使密度矩阵对角化，将系统态驱动为统计混合态。式 (148) 实际上是热态的密度矩阵，两个模中平均共有 n_k 个粒子。由于它完全对角，因此被认为是完全退相干过程的结果。非常重要的一点是，矩阵的 (非) 对角性依赖于基的选择，例如该矩阵原本在双模压缩态基下就是对角的。退相干使密度矩阵对角化的基被称为指针基。我们再次看到，选择哪个基和算符来分析系统态是至关重要的。对于宇宙学微扰，人们已经考虑了多种指针基: 相干态基 [44, 109]、场振幅基 [106, 107, 110, 111]、粒子数基 [110] 以及其他基 [112]。请注意，其中部分工作 [110, 111] 完成于上一节提及的那些研究之前。事实上，在“退相干而不衰相干”的论点提出之前，人们就已经在早期宇宙背景下研究过退相干了。它尤其被用于尝试理解量子宇宙学解的意义，在量子宇宙学中背景和微扰都被当作量子场处理 [113]。归根结底，在 realistic 模型中，指针基由所选相互作用哈密顿量的本征态给出。因此该基具有双重物理意义: 它告诉我们系统对哪类测量表现出经典性，例如场振幅测量或粒子数测量，也告诉我们环境对系统的哪些算敏感。在文献 [39] 的后续工作 [106, 107] 中，Kiefer、Lesgourgues、Starobinski、Polarski 组成的作者团队研究了退相干效应。他们认为，正确的指针基应当是场振幅基，理由是纯引力的自相互作用在场基下是定域的，即 $H_{\mathrm{in}} \propto \hat{\mu}^n(\mathbf{x},\eta)\,\hat{\pi}^m(\mathbf{x},\eta)$ 。由于这些相互作用包含在爱因斯坦-希尔伯特作用量中，它们构成了退相干的一个来源，既最小又定义明确。随后，文献 [114, 115] 首次在更现实的退相干模型中考虑了这些相互作用。在该模型中，系统由可观测的长波长模式构成，环境则由其余未被观测的短波长模式构成，这与随机膨胀中的设置类似 [116]。该方法最初是针对标量微扰提出的，后来被推广到张量微扰 [117]。

How is their influence on the state of $\pm\mathbf{k}$ modes concretely accounted for? In Refs. [114, 115] the process is followed in time, rather than assumed to have completed [109, 112, 118], using a master equation. Earlier papers [110, 111] had also used an equivalent formalism, the Feynman-Vernon influence functional, but only in solvable toy models with two scalar fields interacting quadratically. The two formalisms were also used in Ref. [119], using the short-long wavelengths splitting and considering a quartic self-interaction of the scalar field. To derive a master equation, one starts by postulating that the couple system-environment evolves under a Hamiltonian

如何具体说明它们对 $\pm\mathbf{k}$ 模状态的影响？在文献 [114, 115] 中，人们使用主方程，随时间追踪该过程，而非预先假设过程已完成 [109, 112, 118]。更早的文献 [110, 111] 也使用了等价的形式体系，即费曼-弗农影响泛函，但仅用在两个标量场二次型相互作用的可解玩具模型中。文献 [119] 也同时使用了这两种形式体系，采用长短波长拆分的方法，并考虑了标量场的四次自相互作用。推导主方程时，我们首先假设系统-环境耦合体系由一个哈密顿量支配演化

$$\hat{H}_{\mathrm{tot}} = \hat{H} \otimes \hat{\mathbb{I}}_{\mathrm{env}} + \hat{\mathbb{I}} \otimes \hat{H}_{\mathrm{env}} + g\hat{H}_{\mathrm{int}}\,, \tag{149}$$

where the Hamiltonian of interaction is taken to be an integral of a product of operators acting on the system and the environment

其中相互作用哈密顿量定义为作用在系统上的算符与作用在环境上的算符乘积的积分

$$\hat{H}_{\mathrm{int}} = \int \mathrm{d}^3\mathbf{x}\hat{A}(\eta,\mathbf{x}) \otimes \hat{E}(\eta,\mathbf{x})\,. \tag{150}$$

Under certain assumptions, essentially perturbative coupling and a "large" enough environment unperturbed by the action of the system, the von Neumann equation over the full density matrix $\hat{\rho}_{\text{tot}}$ can be reduced to a master equation over the reduced density matrix of the system $\hat{\rho} = \text{tr}_{\text{env}}\,(\hat{\rho}_{\text{tot}})$. Master equations became a standard tool to analyse the decoherence of cosmological perturbations and are very often [108, 114, 115, 120] considered to be of the Lindblad-type, e.g.,

在特定假设下——本质是微扰耦合，且不受系统作用影响的环境“足够大”——全密度矩阵 $\hat{\rho}_{\text{tot}}$ 满足的冯·诺依曼方程，可以约化为系统约化密度矩阵 $\hat{\rho} = \text{tr}_{\text{env}}\,(\hat{\rho}_{\text{tot}})$ 满足的主方程。主方程现已成为分析宇宙学扰动退相干的标准工具，且通常 [108, 114, 115, 120] 认为其属于林德布拉德型，例如:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}\eta} = -i\left[\hat{H}, \hat{\rho}\right] - g^2\eta_{\mathrm{c}} \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y} \left\langle \hat{E}(\eta, \mathbf{x}) \hat{E}(\eta, \mathbf{y}) \right\rangle \left[\hat{A}(\mathbf{x}), \left[\hat{A}(\mathbf{y}), \hat{\rho}\right]\right], \tag{151}$$

where η_{c} is the auto-correlation time of the environment. This is a Markovian master equation; it assumes that the environment is effectively stationary with respect to the system, i.e., $\eta_{\mathrm{c}} \ll \delta\eta$ where $\delta\eta$ is the typical time-scale of evolution of the system. In addition, the interaction term is often considered linear in the system field operators $H_{\text{int}} \propto (\alpha\hat{\mu} + \beta\hat{\pi}) \otimes \hat{O}_{\text{env}}$, where $\hat{O}_{\text{env}}$ acts only on the environment [39, 120]. It is the so-called Caldeira-Legget model [121]. Such interactions can also be identified as the dominant term when considering pure gravity [114, 117] and has the great advantage of preserving Gaussianity and homogeneity. The result of the evolution can therefore be simply analysed by considering a Gaussian decohered homogeneous density matrix (GHDM). This class of state was introduced in Refs. [44, 122] to study decoherence finely, without having to assume any specific master equation and still preserving a "partially" decohered state rather than assuming from the onset the density matrix diagonal. This class also encompasses the density matrices obtained by the common ansatz that its non-diagonal terms are suppressed by a Gaussian as, e.g., in Refs. [100, 123]. For all these reasons, we will in this section analyse the effect of decoherence using the GHDM and follow Refs. [44,46].

其中 η_{c} 是环境的自关联时间。这是一个马尔可夫主方程；它假设环境相对于系统是有效平稳的，即 $\eta_{\mathrm{c}} \ll \delta\eta$，其中 $\delta\eta$ 是系统演化的典型时间尺度。此外，相互作用项通常被认为是系统场算符 $H_{\text{int}} \propto (\alpha\hat{\mu} + \beta\hat{\pi}) \otimes \hat{O}_{\text{env}}$ 的线性项，$\hat{O}_{\text{env}}$ 仅作用在环境上 [39, 120]。这就是所谓的卡尔德拉-莱格特模型 [121]。这类相互作用在纯引力情形中也被证明是主导项 [114, 117]，且其一大优势是可以保留高斯性与均匀性。因此演化结果可以通过分析高斯退相干均匀密度矩阵 (GHDM) 简单得到。这类态由文献 [44, 122] 引入，目的是细致研究退相干，无需预先假设任何特定主方程，仍能保留“部分”退相干态，而非从一开始就假设密度矩阵是对角的。这类态也包含了由常用近似得到的密度矩阵: 该近似认为非对角项会被高斯因子压制，例如文献 [100, 123] 中的做法。基于上述原因，我们将在本节沿用文献 [44,46] 的思路，利用 GHDM 分析退相干的效应。

To define the GHDM, we work in Fourier space. First, to avoid a preferred direction, all one-point correlation functions have to vanish. The Gaussian state is then completely characterised by its covariance matrix (58) made of two-point correlation functions. By homogeneity, the only non-vanishing two-point correlation functions involve $\mathbf{k}$ and $-\mathbf{k}$, and we can work with a single couple of modes $\pm\mathbf{k}$. A priori, we have a 4×4 matrix, but, as mentioned below Eq. (63), homogeneity further imposes that the matrix is block diagonal in the R/I partition. We are left with a 2×2 covariance matrix like that of Eq. (64). The state is then fully characterised by the three real covariance matrix elements γ_{ij} in Eq. (65) or alternatively the number of pairs n_k and their pair correlation c_k (one complex and one real number) defined in Eq. (78). The only difference with the previous analyses is that the constraint imposed by the purity of the state $p_k = 1$ is now relaxed to $p_k \leq 1$, i.e., $\det\left(\gamma^{\mathrm{S}}\right) = \gamma_{11}\gamma_{22} - \gamma_{12}^2 \geq 1$ or, equivalently $|c_k| \leq \sqrt{n_k\left(n_k+1\right)}$. Notice that these numbers can still not be arbitrarily chosen in order to keep a bona fide quantum state with purity bounded by one. Finally, to be able to have a simple geometrical representation, we can use the purity as an effective extra squeezing parameter and write [46]

为定义 GHDM，我们在傅里叶空间开展计算。首先，为避免存在优先方向，所有一点关联函数必须为零。高斯态完全由两点关联函数构成的协方差矩阵 (58) 描述。根据均匀性，仅存的非零两点关联函数涉及 $\mathbf{k}$ 和 $-\mathbf{k}$ ，因此我们可以仅对一对模 $\pm\mathbf{k}$ 进行计算。原则上我们得到一个 4×4 矩阵，但正如式 (63) 下方所述，均匀性进一步要求该矩阵在实部/虚部分块下是分块对角的。最终我们得到如式 (64) 所示的 2×2 协方差矩阵。该态完全由式 (65) 中的三个实协方差矩阵元 γ_{ij} 描述，也可由式 (78) 定义的对数目 n_k 和对关联 c_k (一个复数和一个实数) 描述。与先前分析的唯一区别是，态纯度施加的约束 $p_k = 1$ 现在放宽为 $p_k \leq 1$ ，即 $\det\left(\gamma^{\mathrm{S}}\right) = \gamma_{11}\gamma_{22} - \gamma_{12}^2 \geq 1$ ，等价于 $|c_k| \leq \sqrt{n_k\left(n_k+1\right)}$ 。注意这些参数仍不能任意选取，以保证得到一个纯度不超过 1 的合法量子态。最后，为得到简洁的几何表示，我们可以将纯度用作一个额外的有效压缩参数，写作 [46]

$$\gamma_{11} = p_k^{-1/2}\left[\cosh\left(2r_k\right) - \cos\left(2\varphi_k\right)\sinh\left(2r_k\right)\right], \tag{152a}$$

$$\gamma_{12} = \gamma_{21} = -p_k^{-1/2}\sin\left(2\varphi_k\right)\sinh\left(2r_k\right), \tag{152b}$$

$$\gamma_{22} = p_k^{-1/2}\left[\cosh\left(2r_k\right) + \cos\left(2\varphi_k\right)\sinh\left(2r_k\right)\right]. \tag{152c}$$

One can check that this is a fully general parametrisation of a 2×2 symmetric matrix that indeed $\det(\gamma) = p_k^{-2}$ and that for $p_k = 1$, we recover Eq. (66). How is the geometrical representation affected by this additional parameter? It is readily seen that the eigenvectors of γ are unchanged, and its eigenvalues simply increased by $p_k^{-1/2} \geq 1$. The effect on the $\sqrt{2}-\sigma$ contour levels is thus simply a dilation by $p_k^{-1/4}$. This increased width of the Gaussian was already noticed as an effect of decoherence in Ref. [123] and before in a different context by Ref. [124]. An important remark is that the existence of a sub-fluctuant mode due to squeezing is not guaranteed anymore since the semi-minor axis is now of length $B_k = p_k^{-1/4}e^{-r_k}$, which can always be made larger than one, the vacuum value, provided that decoherence is strong enough at a given value of squeezing r_k . Figure 6 illustrates the ellipse corresponding to the state in Fig. 3 after having lost purity to $p_k = 0.17$; there is no sub-fluctuant direction. We mention an alternative parametrisation, used in Refs. [44, 88], where the extent of the breaking of the relation between n_k and $|c_k|$ is used to interpolate between a two-mode squeezed state and a thermal state at fixed n_k . We define δ_k such that

可以验证，这是 2×2 对称矩阵的完全通用参数化，满足 $\det(\gamma) = p_k^{-2}$ ，且当 $p_k = 1$ 时，我们可以回推出式 (66)。这个额外参数会对几何表示产生什么影响？不难看出，γ 的本征向量保持不变，仅本征值增加了 $p_k^{-1/2} \geq 1$ 。因此它对 $\sqrt{2}-\sigma$ 轮廓线的作用仅为按 $p_k^{-1/4}$ 做缩放。高斯分布宽度增加作为退相干的效应早已在文献 [123] 中被注意到，更早之前文献 [124] 也在不同背景下指出了这一点。一个重要结论是，压缩产生亚涨落模的性质不再成立，因为半短轴现在长度为 $B_k = p_k^{-1/4}e^{-r_k}$ ，只要给定压缩量 r_k 时退相干足够强，半短轴长度总能大于真空值 1。图 6 展示了图 3 中的态在纯度降低至 $p_k = 0.17$ 后对应的椭圆；该态不存在亚涨落方向。我们在此介绍文献 [44, 88] 中使用的另一种参数化方法: 该方法用 n_k 与 $|c_k|$ 之间关系的破缺程度，在固定 n_k 下对双模压缩态和热态进行插值。我们定义 δ_k 满足

$$|c_k| = (n_k + 1)(n_k - \delta_k). \tag{153}$$

The case $\delta_k = 0$ is a two-mode squeezed state and $\delta_k = n_k$, the maximal value, is a thermal state. This parameter is easily related to the purity and the squeezing via

当 $\delta_k = 0$ 时，该态为双模压缩态；当 $\delta_k = n_k$ 取最大值时，该态为热态。这个参数可以很容易通过纯度和压缩量表示为

$$\delta_k = \frac{1}{2\sqrt{p_k}} \frac{1 - p_k}{\cosh(2r_k) + \sqrt{p_k}}. \tag{154}$$

Let us investigate the effect of decoherence using this class of state. To start with, how is the level of decoherence of the state estimated? Several criteria have been used in the literature: the so-called rate of de-separation [106], evaluating the suppression of non-diagonal terms [114, 115, 123], the positivity time if the initial state is assumed to be non-Gaussian [108], δ_k [44] or simply the purity p_k [120]. We will use the latter since it directly enters our definition (152) of GHDM. The purity can also be conveniently related to the entropy by Eq. (128), which still applies for decohered states. Since the purity has decreased, the entropy increases and becomes non-vanishing. For instance, a thermal state in the two-mode particle number basis (148) gives $c_k = r_k = 0$ and $p_k = (2n_k + 1)^{-1}$.

让我们利用这类态研究退相干的效应。首先，如何估算该态的退相干程度？文献中已经使用了若干判据: 所谓的退分离率 [106]，对非对角项抑制的评估 [114, 115, 123]，初始态假设为非高斯态时的正性时间 [108]、δ_k [44]，或者直接用纯度 p_k [120]。我们将采用最后一种，因为它直接进入我们对 GHDM 的定义 (152)。纯度还可以通过式 (128) 方便地与熵联系起来，该式对退相干态仍然成立。由于纯度降低，熵会升高并变为非零。例如，双模粒子数基矢 (148) 下的热态给出 $c_k = r_k = 0$ 和 $p_k = (2n_k + 1)^{-1}$。

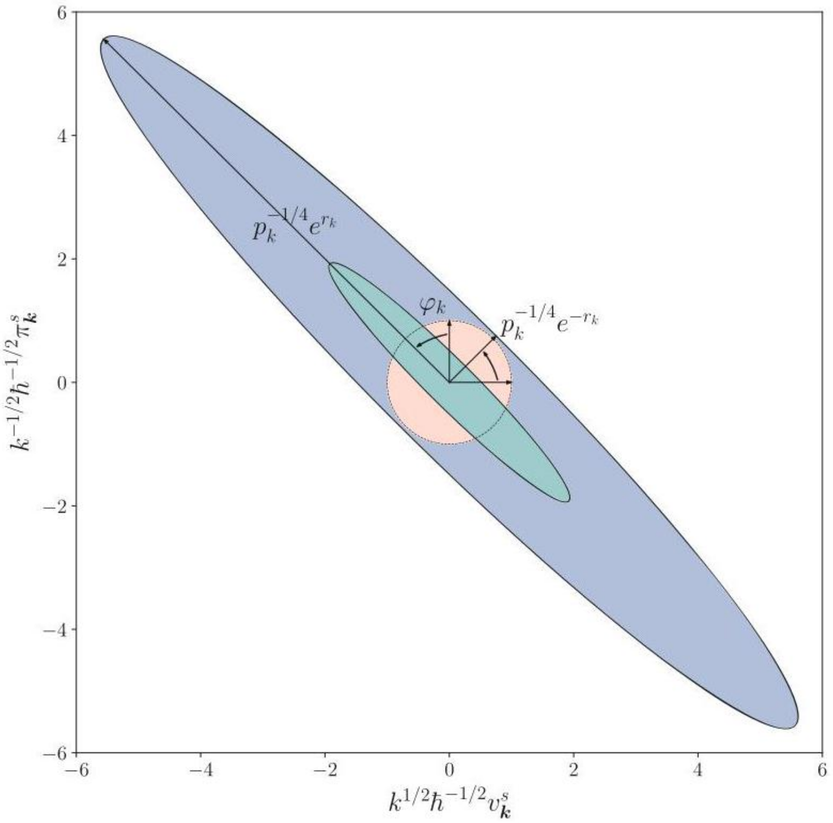
Our focus is on how a certain level of decoherence, represented by p_k, can lead to a classical state in the sense of the criteria discussed in the previous section. As we now show, for mixed states, the different criteria are, in general, inequivalent and give different answers [122]. The separability condition Eq. (131) is also still valid for the partially decohered distribution [44]. It has a very elegant interpretation when rewritten in terms of the effective squeezing parameter

我们的重点是，由 p_k 表征的一定程度的退相干，如何按照上一节讨论的判据得到经典态。下文我们将说明，对于混合态，不同判据通常不对等，会给出不同结果 [122]。可分性条件式 (131) 对部分退相干的分布仍然成立 [44]。当用有效压缩参数改写时，它有一个非常简洁的诠释

$$B_k \hbar^{-1/2} = p_k^{-1/4} e^{-r_k} \leq 1, \tag{155}$$

Fig. $6\sqrt{2} - \sigma$ contour level of the Wigner function W^{S} for $\varphi_k = \pi/4, r_k = 1, p_k = 0.12$ (blue ellipse) or $p_k = 1$ (green ellipse) and the vacuum state $r_k = 0$ (pink circle)

图 $6\sqrt{2} - \sigma$ 为 Wigner 函数 W^{S} 的等高线，对应 $\varphi_k = \pi/4, r_k = 1, p_k = 0.12$ (蓝色椭圆) 或 $p_k = 1$ (绿色椭圆)，以及真空态 $r_k = 0$ (粉色圆形)



i.e., the state becomes separable when there is no sub-fluctuant mode anymore due to a sufficient level of decoherence $p_k < e^{r_k}$. The condition can also be written as $\delta_k \geq n_k/(n_k+1)$ which, for the very large number of primordial gravitons expected $n_k \gg 1$, becomes $\delta_k \geq 1$ [44].

也就是说，当达到足够的退相干程度 $p_k < e^{r_k}$ 后，不再存在亚涨落模式，态就变成可分态。该条件也可以写成 $\delta_k \geq n_k/(n_k+1)$ ，对于预期的原初引力子大数量 $n_k \gg 1$ ，条件变为 $\delta_k \geq 1$ [44]。

Let us now turn to the Bell inequality of Eq. (141). Its form, its maximisation procedure and the formula Eq. (139) are still valid for our partially decohered state. We plot the value of $C_{\max}$ for a modest number of gravitons in each polarisation $n_k = 100$ and different values of δ_k in Fig. 7. We see that the maximum of $C_{\max}$ gradually recedes away from violation as δ_k increases and that for $\delta_k = 0.1$, the inequality is not violated anymore. In Ref. [88], the authors give an approximation in the limit $\delta_k \ll n_k$, which is equivalent to $\cosh^2(r_k) \gg 1$, i.e., in the limit of a very squeezed state. In this limit, we have

现在我们来看式 (141) 的贝尔不等式。它的形式、最大化过程以及式 (139) 对我们的部分退相干态仍然成立。我们在图 7 中绘制了每个偏振 $n_k = 100$ 取 modest 引力子数量、δ_k 取不同值时 $C_{\max}$ 的取值。我们可以看到，随着 δ_k 增大，$C_{\max}$ 的最大值逐渐远离违反区间，当取 $\delta_k = 0.1$ 时，不等式不再被违反。文献 [88] 中，作者在极限 $\delta_k \ll n_k$ 下给出了一个近似，该近似等价于 $\cosh^2(r_k) \gg 1$ ，也就是极高压缩态的极限。在该极限下，我们有

$$C_{\max}\left(|v|^2\right) = \frac{1}{2(1+\delta_k)}\left[1 + \frac{3}{2^{4/3}} + O\left(\frac{1+\delta_k}{n_k}\right)\right]. \tag{156}$$

so that inequality is violated when

因此当满足下述条件时，不等式被违反

$$\delta_k < 0.095\,. \tag{157}$$

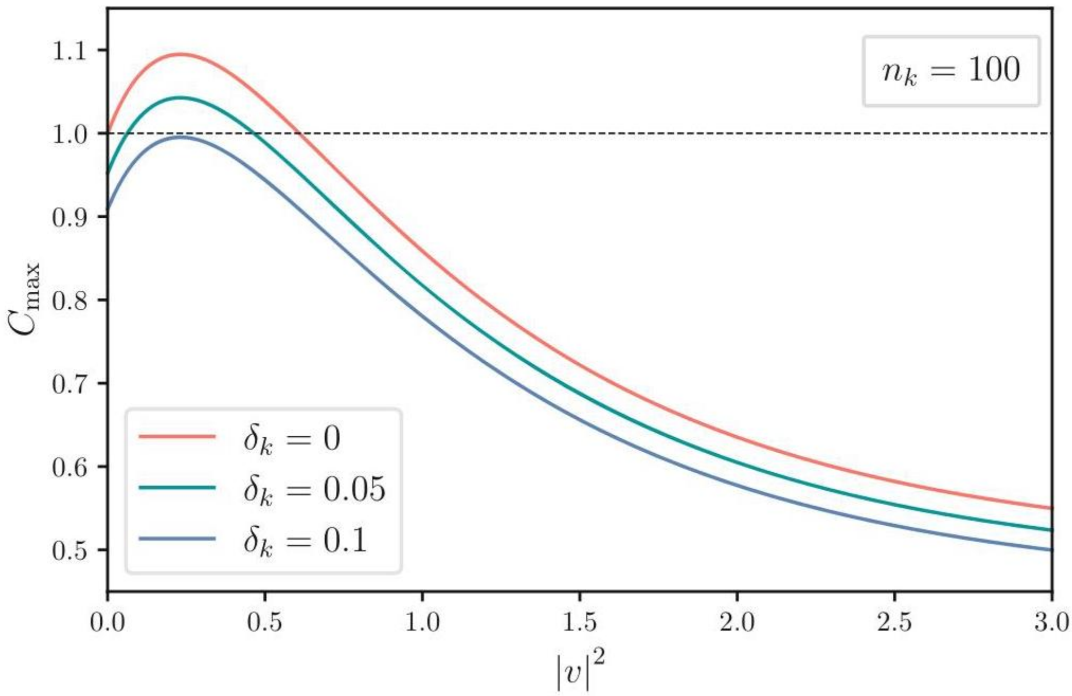


Fig. 7 $C_{\max}$ as a function of $|v|^2$ is shown in full line for different values of δ_k . The threshold of Bell inequality violation $C_{\max} = 1$ is shown in dashed black line

图 7 $C_{\max}$ 为关于 $|v|^2$ 的函数，不同取值的 δ_k 以实线展示。贝尔不等式违反的阈值 $C_{\max} = 1$ 以黑色虚线展示

The threshold is an order of magnitude smaller than that of separability. This condition is unfortunately not easily expressed in a comparison between p_k and r_k . The perturbations loose their quantum character in the sense of the Bell inequality Eq. (141) faster than in the sense of separability. This is expected since we recall that separability is a necessary condition for Bell inequality violation, and here we see that it is not a sufficient condition anymore; the criteria are inequivalent for mixed states.

该阈值比可分性阈值小一个数量级。遗憾的是，这个条件无法简单地通过 p_k 和 r_k 的对比表示出来。扰动在式 (141) 贝尔不等式的意义下比在可分性意义下更快失去量子特性。这符合预期，因为我们知道可分性是贝尔不等式被违反的必要条件，而我们在此看到它不再是充分条件；对于混合态，不同判据不对等。

Finally, let us examine the behaviour of the quantum discord. The formula (146) was generalised in Ref. [46] for partially decohered states. A similar computation in presence of decoherence, although less general, was previously carried out in Ref. [125]. The generalisation reads

最后，我们来研究量子失协的行为。文献 [46] 已将公式 (146) 推广到部分退相干态。此前文献 [125] 中也开展过类似计算，该计算存在退相干但适用范围更窄。推广后的公式为

$$\mathcal{D}_{\pm\mathbf{k}} = f\left[p_k^{-1/2}\cosh(2r_k)\right] - 2f\left(p_k^{-1}\right) + f\left[\frac{p_k^{-1/2}\cosh(2r_k) + p_k^{-1}}{p_k^{-1/2}\cosh(2r_k) + 1}\right]. \tag{158}$$

One notes that the discord does not depend on the squeezing angle φ_k. This angle can always be modified by a local symplectic transformation, and the discord is a local symplectic invariant, so it must not depend on it. In Fig. 8, we plot this formula as a function of p_k and r_k and draw the line delimiting separable from non-separable states. Its complexity prevents us from giving a simple threshold for the discord to be, say, larger than 1 and to compare with separability and Bell inequality. Figure 8 shows clearly that, as for separability, the value of the discord is dictated by the result of a competition between the level of squeezing r_k and that of decoherence p_k. These two criteria, along with a Bell inequality of the type considered in Ref. [100], were recently compared in Ref. [126].

可以看到，量子失协不依赖压缩角 φ_k。压缩角总能通过局部辛变换改变，而量子失协是局部辛不变量，因此它必然不依赖压缩角。在图 8 中，我们将该公式绘制成关于 p_k 和 r_k 的函数，并绘制出划分可分态与不可分态的分界线。该公式的复杂性导致我们无法给出量子失协大于 1 这类简单阈值，也无法与可分性、贝尔不等式进行比较。图 8 清楚表明，和可分性一样，量子失协的值由压缩水平 r_k 和退相干水平 p_k 共同竞争决定。近期文献 [126] 已将这两项判据与文献 [100] 中研究的某类贝尔不等式进行了比较。

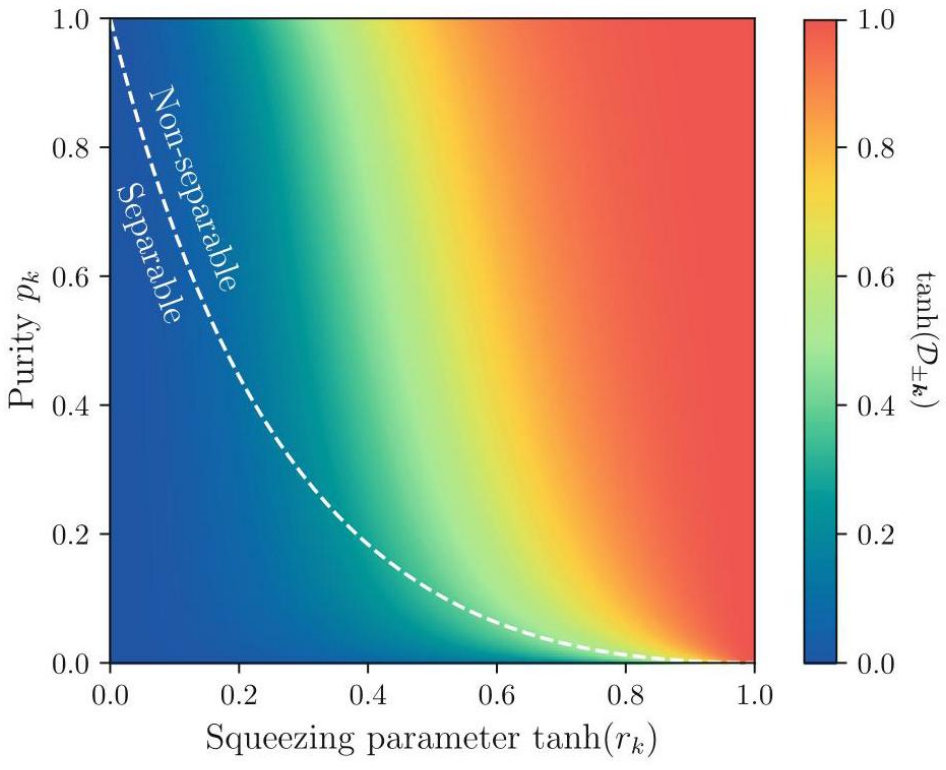


Fig. 8 Quantum discord $\mathcal{D}_{\pm k}$ of Eq. (158) for a partially decohered state defined by Eq. (152) as a function of its squeezing r_k and a purity p_k

图 8 部分退相干态由式 (152) 定义，其式 (158) 的量子失协 $\mathcal{D}_{\pm k}$ 随压缩参数 r_k 和纯度 p_k 变化的函数图

The overall result of this discussion is that decoherence, if large enough, does, in the sense of different inequivalent criteria, erase the quantum features of the state. To be able to complete the analysis, the only thing necessary is to get a realistic estimate of the loss of purity in the early universe. Can we get observational constraints on the interactions generating decoherence and on its level? Unfortunately, not for primordial gravitational waves since they were not detected yet. However, for scalar perturbations, the observation of the baryonic acoustic oscillation (BAO) actually imposes that during inflation, decoherence cannot modify too much the squeezing parameters [44] $r_k \gg 1$ and $\varphi_k \approx -\pi/2$. In particular, this implies that complete decoherence during inflation, leading to a thermal state like Eq. (148), is excluded. Indeed, the squeezing parameter r_k would vanish [106]. Note that the purity p_k is not constrained by this argument. This relation between the oscillations and strong squeezing had initially led to label the former a quantum feature [30]. As we have explained, the squeezing, in its dynamical aspect, can be understood as the presence of a growing and a decaying mode so that this result can be understood completely classically as pointed in Ref. [45]. This ”temporal coherence” of the perturbations is explained in detail (using a classical point of view) in Ref. [82]. In addition of this general argument, for precise models of decoherence, other constraints can be obtained as discussed, for instance, in Ref. [120].

上述讨论的总体结论是: 根据多个不等价判据，当退相干强度足够大时，它会抹除该态的量子特性。要完成分析，唯一需要做的就是对早期宇宙的纯度损失给出符合实际的估计。我们能否对产生退相干的相互作用及其强度给出观测限制? 很遗憾，针对原初引力波暂时无法做到，因为我们尚未探测到原初引力波。但对于标量扰动，对重子声学振荡 (BAO) 的观测实际上要求: 暴胀过程中，退相干不能过度改变压缩参数 [44] $r_k \gg 1$ and $\varphi_k \approx -\pi/2$ 。这尤其说明，暴胀过程中形成式 (148) 那样热态的完全退相干是不成立的，因为此时压缩参数 r_k 会变为零 [106]。注意，该论证并未对纯度 p_k 给出限制。重子声学振荡与强压缩之间的这种关联最初让人们将前者归为量子特性 [30]。但正如我们此前解释的，从动力学角度看，压缩可以理解为存在一个增长模式和一个衰减模式，因此正如文献 [45] 指出的，该结果完全可以从经典角度解释。文献 [82] 从经典视角详细解释了扰动的这种“时间相干性”。除了这个一般性论证，正如文献 [120] 中讨论的，针对具体退相干模型还可以得到其他限制条件。

Let us close this section by coming back to the important question of the observability of the features. Even in the absence of decoherence, are the operators that we have used in the discussion measurable? For the Bell inequality (141) they have derived in Ref. [88], Campo and Parentani argued that each of the four terms is, in principle, measurable. However, one needs to measure a difference of order 1 between these while the intrinsic fluctuation of the factor n_k is of order n_k , which is of order 10^{86} . The measure is, in practice, impossible. The authors of Ref. [100] argue that having only access to the growing mode makes it impossible to measure two of their three pseudo-spin operators. Verifying their Bell inequality necessitates measuring at least two, and so is experimentally impossible. To address this difficulty, they suggest that one could try to build Legget-Garg inequalities [127] that rely on correlation in time of a single operator and do not require to measure two non-commuting operators at a given time. Ref. [128] also proposed a ”baroque”, to use the term of the author, inflationary model in which Bell operators are measured during inflation by another field rather than at later times by observer. The field stores the result in classical, robust variables that could be read out at later times by observers. Finally, the separability and quantum discord, being directly attached to properties of the density matrix, seem harder to measure. The possibility of measuring them directly in the cosmological case has not, to the best of our knowledge, been analysed. In Ref. [37], the authors took another approach and showed that if the perturbations were in a quantum non-discordant state and reproduced the power spectrum measured for scalar perturbations, then they have to be in the thermal state (148). As we have just explained, this is ruled out. Note that this argument assumes that the system is described by a quantum state rather than proves it.

最后我们回到关于这些特征可观测性的重要问题来结束本节。即便不存在退相干，我们讨论中用到的算符是可测量的吗？针对文献 [88] 中推导得出的贝尔不等式 (141)，Campo 和 Parentani 认为四个项中的每一个原则上都是可测量的。然而，我们需要测量这些项之间量级为 1 的差值，而因子 n_k 的本征涨落量级为 n_k，即量级为 10^{86}。实际上该测量是不可能实现的。文献 [100] 的作者指出，仅能获取增长模式意味着无法测量其三个赝自旋算符中的两个。验证他们的贝尔不等式需要至少测量两个算符，因此在实验上不可能实现。为解决这一难点，他们提出可以尝试构建莱格特-加格不等式 [127]，这类不等式依赖于单个算符的时间关联，不需要在同一时刻测量两个不对易算符。文献 [128] 也提出了一个用作者自己的话来说“复杂离奇”的暴胀模型: 贝尔算符不是在后期由观测者测量，而是在暴胀过程中由另一个场完成测量，该场将结果存储在稳定的经典变量中，供观测者在后期读取。最后，可分性和量子失谐直接和密度矩阵的性质绑定，似乎更难测量。据我们所知，在宇宙学场景中直接测量它们的可能性尚未得到分析。文献 [37] 的作者采用了另一种方法，他们证明如果扰动处于量子无失谐态，并且可以重现观测到的标量扰动功率谱，那么它们一定处于热态 (148)。正如我们刚才解释的，这一情况已经被排除。需要注意的是，这个论证假设系统由量子态描述，而非证明系统本身就是量子态。

Some Perspectives and Critics

若干视角与批评

To conclude this review, we mention a few perspectives and possible criticisms of the previously developed issues.

在本综述的最后，我们针对前文讨论的内容提出若干视角与可能的批评。

First, the estimation of the minimal level of decoherence of cosmological perturbations keeps being refined; see, e.g., Refs. [117, 129, 130]. Most authors conclude that decoherence has completed by the end of inflation, and the state is classical when the modes become sub-Hubble again. However, an application of the precise level of decoherence obtained to a concrete non-classicality criterion is still missing. Such computation would be essential since we have shown that the threshold for the emergence of classicality given by the different criteria depends on both the purity and the level of squeezing. In addition, some authors have also suggested that the use of Markovian approximation is not well-justified in the cosmological context and that a more general master equation is required to achieve a correct prediction [131, 132].

第一，宇宙学扰动退相干最小程度的估算仍在不断精细化；参见例如文献 [117, 129, 130]。大多数学者得出结论: 退相干在暴胀结束时已完成，当模式再次成为哈勃尺度内模式时，其状态已成为经典态。然而，目前仍未将得到的精确退相干程度应用到具体的非经典性判据中。这类计算是十分必要的，因为我们已经证明，不同判据给出的经典性涌现阈值同时依赖于纯度和压缩程度。此外，也有学者指出，在宇宙学背景下，马尔可夫近似的使用缺乏充分依据，要得到正确的预言需要使用更广义的主方程 [131, 132]。

Second, the discussion of section "Quantum Features in Primordial Gravitational Waves?" applies to the tensor and scalar perturbations. However, primordial gravitational waves have the important specificity that they could be directly detected, not only indirectly in the temperature anisotropies of the CMB, as scalar ones. Direct detection (although futuristic [73]) would bring about exciting possibilities to search for quantum signatures in gravitational wave detectors. Several authors, e.g., [133-135], have investigated these. The squeezed states of gravitons could produce noise in gravitational wave interferometers, and some of the authors argued that its quantum character might be revealed by measuring the decoherence it would induce between two entangled mirrors.

第二，"原初引力波中的量子特性?"一节的讨论同时适用于张量扰动和标量扰动。但原初引力波有一个重要特点: 它可以被直接探测，而不像标量扰动那样只能通过宇宙微波背景的温度各向异性间接探测。直接探测 (尽管还属于未来规划 [73]) 将为在引力波探测器中寻找量子 signatures 带来令人振奋的可能，已有多位学者对此展开研究，例如 [133-135]。引力子的压缩态会在引力波干涉仪中产生噪声，部分学者提出，通过测量它在两个纠缠反射镜之间诱导的退相干，可以揭示其量子特性。

Another possibility that we have not discussed is to use the interactions of the perturbations, not as a mere source of decoherence, but as giving new signals in the form of non-Gaussianities that could be used. Focusing on scalar perturbations, the authors of Ref. [136] showed that substituting the initial quantum vacuum fluctuations by a Gaussian stochastic field with the same two-point functions would lead to enhanced non-gaussianities akin to those generated by initial excited states. Not measuring such an enhancement was then suggested to be a sign of nonclassicality of the initial state (see also Ref. [137]). With a different approach to non-Gaussianities, the Wigner function of primordial gravitational waves was calculated in Ref. [138], taking in account the intrinsic non-linearities of gravity. Its regions of negativity were then explored as a means of exhibiting a signature of quantumness of the state. Other works such as Refs. [139, 140] took yet another route and provided some constraints on decoherence based on the level of non-Gaussianities.

我们尚未讨论的另一个研究方向是: 不将扰动的相互作用仅作为退相干的来源，而是利用这类相互作用得到非高斯性形式的新信号。针对标量扰动，文献 [136] 的作者证明，若将初始量子真空涨落替换为具有相同二点关联函数的高斯随机场，会得到和初始激发态类似的增强非高斯性。因此有观点认为，未观测到这类增强是初始态具有非经典性的标志 (另见文献 [137])。文献 [138] 采用了不同的非高斯性研究方法，在考虑引力内禀非线性的情况下计算了原初引力波的维格纳函数，进而通过分析维格纳函数的负区域来寻找该态量子性的 signature。文献 [139, 140] 等其他工作则采用了另一条路径，基于非高斯性的水平给出了退相干的若干限制。

Finally, some authors criticised the standard approach of analysing correlations between $\pm\mathbf{k}$ modes. The authors of Refs. [122,141] have argued that discussing correlations between $\pm\mathbf{k}$ modes is not appropriate as these two modes do not exist separately outside of Minkowski, in particular during inflation, and keep being mixed. Just as there is no preferred choice of vacuum (Section "Particle Production"), there is no preferred choice of partition to unambiguously discuss levels of squeezing and correlations. These critics, we believe, would not apply to sub-Hubble modes, e.g., in our toy model radiation domination where $a' = 0$. Some recent works [142, 143] do not suffer from these shortcomings since they perform similar computations for quantum discord and Bell inequalities but use real space correlation functions. Unfortunately, their results tend to show that, even in the absence of decoherence, no quantum features appear in real space. Lastly, the formalism presented here does not address the so-called quantum measurement problem in cosmology. In our approach, we used an ergodicity assumption to justify equating the quantum expectation values to average values over different patches of the sky. However, one could argue that we did not discuss how the perturbations "collapsed" from a homogeneous quantum state to an inhomogeneous distribution with different values in each patch. For a discussion of this point, see Ref. [144].

最后，有学者批评了分析 $\pm\mathbf{k}$ 模式之间关联的标准方法。文献 [122, 141] 的作者提出，讨论 $\pm\mathbf{k}$ 模式之间的关联并不合适，因为这两个模式并非在闵氏空间之外 (尤其是暴胀期间) 独立存在，而是始终保持混合。正如不存在优先的真空选择 (“粒子产生”一节)，也不存在优先的分割方式来清晰讨论压缩程度和关联水平。我们认为，这类批评不适用于哈勃尺度内模式，例如我们的 toy model 辐射主导阶段中 $a' = 0$。近期的若干工作 [142, 143] 不存在这些缺点，它们对量子失谐和贝尔不等式也做了类似计算，但采用的是实空间关联函数。遗憾的是，其结果往往表明，即使不存在退相干，实空间中也不会出现量子特性。最后，本文提出的形式体系并未解决宇宙学中所谓的量子测量问题。在我们的方法中，我们使用各态历经假设来证明将量子期望值等同于天区不同 patches 的平均是合理的。但仍可以提出质疑: 我们并未讨论扰动是如何从均匀量子态“坍缩”为每个 patch 具有不同取值的非均匀分布的。关于这一点的讨论，参见文献 [144]。

To conclude, it is fair to say that the current status regarding the quest for quantum features in the primordial gravitational wave background is not entirely settled. First, on the observational side, the waves themselves, even in their classical aspects, have yet to be detected [145]. Experiments in preparation [75, 76] might manage to detect signatures of the waves in the $\mathbf{B}$ -modes of the CMB. However, direct detection via gravitational wave interferometers seems so far out of reach [73]. On the theoretical side, in recent years, several quantum features of the quantum state for the primordial gravitational waves predicted in the simplest models have been exhibited. Unfortunately, no currently available experimental protocol has yet been designed to detect these features. In addition, the effect of decoherence has been increasingly more precisely characterised, and the latest findings tend to show that it might have erased all the potentially detectable features by the end of inflation. At this time, most analyses have been restricted to the simplest inflationary models and at the Gaussian level. More recently, some promising suggestions and proposals have been made concerning non-Gaussianities, discussing the possible signatures of decoherence or other possible hints of a quantum origin of the perturbations.

总而言之，可以公允地说，目前我们对原初引力波背景量子特征的探索尚未完全定论。首先，在观测层面，原初引力波哪怕是其经典层面的性质都仍未被探测到 [145]。筹备中的实验 [75, 76] 或许能够在 CMB 的 $\mathbf{B}$ 模中探测到原初引力波的信号。然而，通过引力波干涉仪直接探测目前看来还遥不可及 [73]。在理论层面，近年来，最简模型预测的原初引力波量子态的若干量子特征已经被揭示出来。遗憾的是，目前还没有设计出可行的实验方案来探测这些特征。此外，退相干效应的刻画精度已经得到了持续提升，最新研究结果往往表明，退相干可能已经在暴胀结束前抹去了所有可被探测的潜在特征。目前，大多数分析都局限于最简暴胀模型和高斯层面。近来，针对非高斯性已经出现了一些颇有前景的提议与讨论，其中探讨了退相干可能留下的信号，或是扰动存在量子起源的其他潜在线索。

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